On the Number of Radial Orderings of Planar Point Sets.

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Abstract

Let S be a set of n points in general position on the plane. For a given point p not in S, a *radial ordering* of S with respect to p is a circular ordering of the elements of S by angle around p. If S is bi-colored, with equal sized color classes, then a *colored radial ordering* of S with respect to p is a circular ordering by angle of the colors of the points of S around p, that is, permutations between points with the same color give the same ordering. The problem studied in this paper is to give bounds on the number of distinct non-colored and colored radial orderings of S. We show that the total number of distinct radial orderings of S is at most $O(n^4)$ and at least $\Omega(n^2)$. In the bi-colored case we show that the number of distinct colored radial orderings of S is at most $O(n^4)$ and at least $\Omega(n)$. An example of a point set with $\Theta(n^4)$ radial orderings is provided for both the non-colored and bi-colored case. In the bi-colored case, we also show how to generate examples with $O(n^2)$ distinct colored radial orderings.

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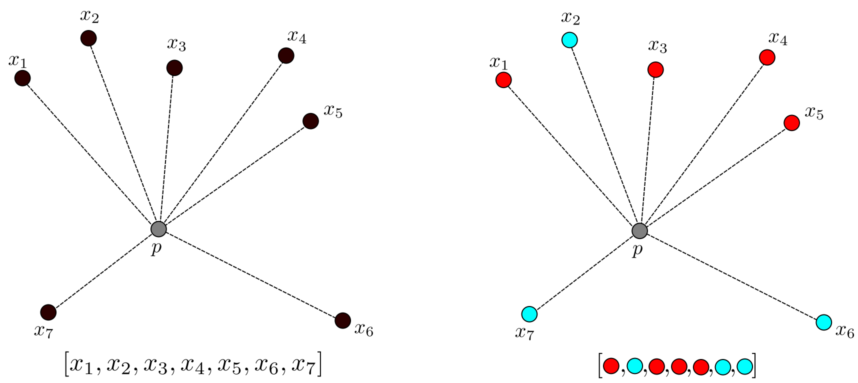

Figure 1: An non-colored and a colored radial ordering.

1 Introduction

Imagine walking around, exploring a city. Although you are always able to see some important buildings, you do not have a map or a compass. The only information available to you is the position of the buildings, relative to one other around you. As you keep walking, their relative positions change. In this paper we are interested in counting the maximum and minimum number of different building configurations that you can see if you visit the whole city. We also consider the case where there are only two classes of buildings and you cannot distinguish between members of the same class.

Formally, let S be a set of n points in general position on the plane and let p be a point not in S, such that $S \cup \{p\}$ is also in general position; we call p an observation point. A *radial ordering* of S with respect to p is a circular clockwise ordering of the elements of S by their angle around p. Thus, if $S = \{x_1, x_2, x_3\}$ for example, then $[x_1, x_2, x_3]$ and $[x_3, x_1, x_2]$ are the same (or equivalent) radial ordering. As mentioned above, we also consider the case when S is partitioned into two sets R and B of the same cardinality (so that n is even). In such a case we think of S as being *colored* and say that the elements of R and B are *red* and *blue*, respectively. In this case, we only care for the color of the elements of S. A *colored radial ordering* with respect to p is the circular ordering of the colors of the points in the radial ordering around p. In other words, permutations between points with the same color give the same colored radial ordering. See Figure 1. Unless otherwise noted, all point sets in this paper are in general position. We are interested in bounding the maximum and minimum number of distinct non-colored and colored radial orderings of S.

A different problem but in the same setting has been studied recently in [9]. In that particular paper, the authors study what a robot can infer from its environment when all the information that is available is the cyclic positions of some landmarks as they appear from the it's position. Other authors have considered problems of the same flavor, when a similar kind of information is available. See for example [5, 6, 8].

We point out that computing the radial ordering of S around every point *in* S is an unavoidable step in some geometric algorithms, as for example, doing a radial sweeping. Moreover, many optimization problems are solved by considering the arrangement of lines passing through two points in S and finding the optimum point inside each of the $O(n^4)$ cells in the arrangement [7]. In many cases this is because the radial ordering of the points in S around every point p within a cell is the same. It could be interesting in this scenario to know how many induce the same radial ordering.

For a bi-colored point set, a radial sweeping algorithm also requires the ordering as an initial step, so it could be useful to know bounds on the number of different colored radial ordering of S from points in the plane. From the combinatorial point of view, this problem is related to partitioning bi-chromatic point sets by using *k-fans* [1, 2]. A k-fan in the plane is a point p (called the center) and k rays emanating from p. This structure can be used to partition the point set S into k monochromatic subsets. The existence and non-existence of balanced k-fans for colored point sets have been studied in recent papers [4, 3] but, as far as we know, the number of different monochromatic partition by k-fans has not yet been considered. At a glance, the number of distinct colored radial orderings seems much smaller than the number of non-colored and the upper bounds appear to be different. Surprisingly, in this paper we provide an example of a bi-colored set of n points with $\Theta(n^4)$ distinct colored radial orderings.

The remainder of the paper is structured as follows. In Section 2 we show that the number of distinct (non-colored) radial orderings of S is always at most $O(n^4)$ and at least $\Omega(n^2)$. In Section 3 we focus on bi-colored point sets. Although the same upper bound $O(n^4)$ applies, the provided lower bound is linear. We give an example of a bi-colored set of n points with $\Theta(n^4)$ colored radial orderings, thus achieving asymptotically the upper bound for both non-colored and colored point sets. In the bi-colored case, we also show how to generate examples with every intermediate value of distinct colored radial orderings. In Section 4 we mention some open problems for future research.

2 Radial Orderings

In this Section we give upper and lower bounds for the number of distinct radial orderings of a (non-colored) set S of n points in the plane. First, we discretize the problem by partitioning the plane into a finite number of regions so that two points in a same element (i.e region) of the partition induce the same radial ordering. The crux of the problem is that the converse is not true; two observation points in different elements of this partition may induce the same radial ordering.

The partition of the observation points is made by half-lines in the plane which if crossed by the observation point, generate a transposition of two consecutive elements in the radial ordering. We call these half lines *swap lines* and define them as follows. For every pair of points x_1 and x_2 in S consider the line passing through them. Contained in this line we have two swap lines; one begins in x_1 and does not contain x_2 while the other begins in x_2 and does not contain x_1. Figure 2 shows the swap lines anchored at x_1 and x_2. Two observation points are in the same element of the partition if they can be connected by a curve which does not intersect any swap lines. We call this partition the *order partition*, and since it induces a decomposition of the plane, we refer to its elements as *cells*. As a set of three points already shows, two points in different cells may induce the same radial ordering, see Figure 3.

Swap lines and the order partition were introduced in [9], where various properties of the order partition were established. The authors refer to the order partition simply as a "decomposition" of the plane.

Note that the order partition readily implies the upper bound on the number distinct of radial orderings of S.

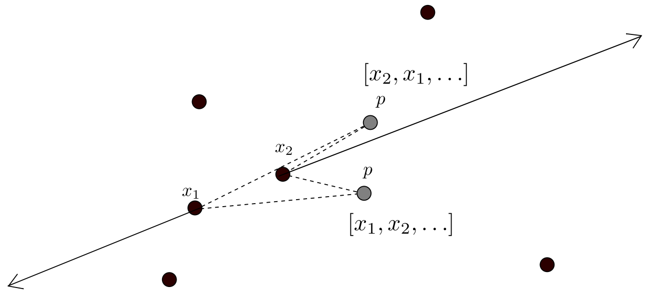

Figure 2: Two swap lines induced by a pair of points of S.

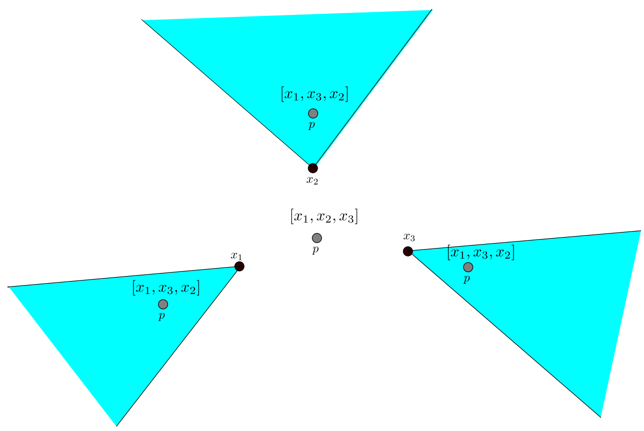

Figure 3: The order partition of three points has four cells. The shadow cells have the same radial ordering.

Theorem 2.1 *The number of distinct radial orderings of S is at most $O(n^4)$.*

Proof. Extend every swap line in S to a straight line. This generates a line arrangement of $\binom{n}{2}$ lines and at most $O(n^4)$ cells. Note that every cell of the order partition is the union of some cells and edges of the line arrangement so constructed. Therefore the order partition cannot have more cells than this line arrangement and the number of distinct radial orderings of S is at most $O(n^4)$. $\square$

The asymptotic upper bound given in Theorem 2.1 is achievable. We provide such an example in Section 3, where we give a colored set with $\Theta(n^4)$ distinct colored radial orderings. Clearly, the number of colored orderings of a point set cannot exceed the number of its (non-colored) radial orderings.

We now prove a general lower bound of $\Omega(n^2)$, by showing a way to "walk" so that $\Omega(n^2)$ different radial orderings are seen.

Theorem 2.2 *The number of distinct radial orderings of S is at least $\Omega(n^2)$.*

Proof. Let ℓ be a straight line having points of S both above and below it. Let p and q be two points in ℓ, such that when walking from p to q in a straight line a swap line is crossed. We show that in this case p and q must induce different radial orderings. This immediately implies that the number of different radial orderings seen when walking along ℓ is equal to the number of intersected swap lines plus one.

Indeed, suppose that the swap line crossed when walking from p to q is defined by two points x_1 and x_2 of S. Note that x_1 and x_2 are both on the same side of the line ℓ. Because of our choice of ℓ there is a point x_3 of S on the side opposite to that of x_1 and x_2. Since the swap line defined by x_1 and x_2 is crossed only once, x_1 and x_2 transpose only once. Furthermore, since x_3 is on the side opposite to x_1 and x_2, then x_3 never transposes x_1 or x_2. This implies that if the radial ordering of $\{x_1, x_2, x_3\}$ around p is $[x_1, x_2, x_3]$, then it would be $[x_1, x_3, x_2]$ around q. Therefore, the radial ordering of S around p is different from the radial ordering around q since they do not agree on $\{x_1, x_2, x_3\}$

It remains to show that there is a choice of ℓ crossing the necessary number of swap lines. Let us choose ℓ to be a line having only one point of S above and all the others below, and such that ℓ is not parallel to any swap line. Note that ℓ intersects every swap line defined by pairs of points of S below ℓ. Since there are $n-1$ such points, they define $\binom{n-1}{2}$ such swap lines and the result thus follows. $\square$

We now turn our attention to the colored case.

3 Colored Radial Orderings

Assume that n is even and that S is partitioned into a red set R and a blue set B of the same size. Since there cannot be more colored than non-colored radial orderings, the upper bound of $O(n^4)$ given in Theorem 2.1 still applies for S. We now provide the promised example achieving this upper bound.

Theorem 3.1 *A positive constant* $\mathbf{c}$ *exists, such that for every natural even number* $n \geq 160$*, a bi-colored set of* n *points can be arranged with at least* $\mathbf{c} \cdot (n-19)^4$ *colored radial orderings.*

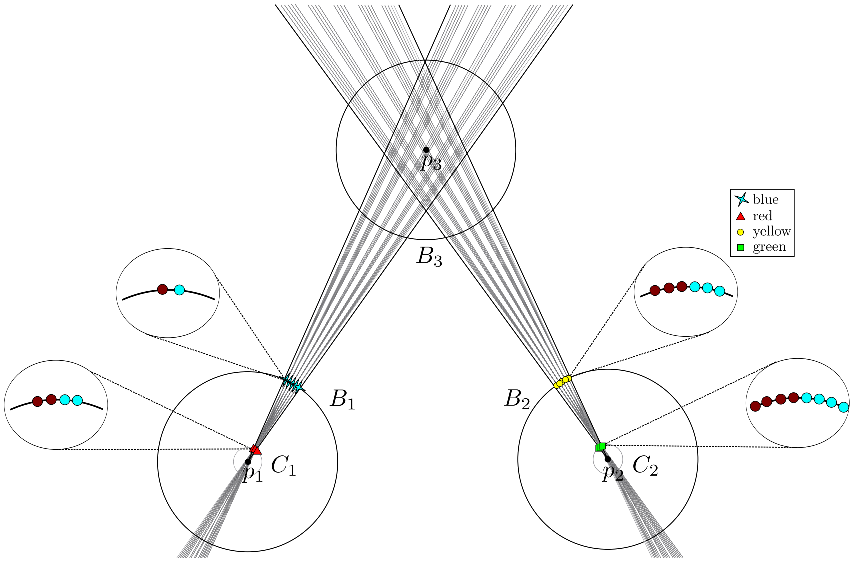

Figure 4: A bi-colored set with $\Omega(n^4)$ different colored radial orderings.

Proof. The general idea of the proof is the following. We start by first constructing a four-colored set of points S' with $\Theta(n^4)$ distinct colored radial orderings; afterwards we replace each point of a given color with a suitable "pattern" of red and blue points. These four patterns are chosen so that if they appear consecutively in a radial ordering, then any other equivalent radial ordering must match patterns of the same type. Since the patterns behave like the original four colors, the new set also has $\Theta(n^4)$ colored radial orderings.

We describe the construction of S' in detail. Let B_1, B_2 and B_3 be three balls of radius $1/4$, whose centers p_1, p_2 and p_3, are the vertices of an equilateral triangle of side length equal to one. Consider small enough values $\varepsilon, \alpha > 0$ (to be detailed later). Let C_1 and C_2 be circles of radius $\varepsilon > 0$ centered at p_1 and p_2, respectively. Let γ_1 and γ_2 be infinite wedges of angle α, with apices p_1 and p_2 respectively. Assume that γ_1 is bisected by the line segment joining p_1 and p_3, while γ_2 is bisected by the line segment joining p_2 and p_3. Refer to Figure 4.

Let m and r be the only natural numbers such that $n = 20m + r$ and $0 \leq r \leq 19$. Since $n \geq 160$, we have that $m \geq 8$. Divide γ_1 evenly with $m-1$ infinite rays centered at p_1 and intersecting B_3, such that the angle between two consecutive rays is α/m. Do likewise for γ_2, with $m-1$ infinite rays centered at p_2 . At every point of intersection of these rays with the boundary of B_1, place a blue point; at every point of intersection with C_1 a red point; at every point of intersection with the boundary of B_2 a yellow point; finally at every point of intersection with C_2 a green point. Note that neither the rays centered at p_1 intersect B_2 nor the rays centered at p_2 intersect B_1. Thus $4(m-1)$ colored points are placed in total.

We choose α and ε to be small enough so that the two following conditions are met: (1) The intersection of γ_1 and γ_2 is contained in the interior of B_3; (2) the point of intersection (if it exists) of every line passing through a red and a blue point with any other line passing through a red and blue point lies in the interior of B_1; (3) the point of intersection (if it exists) of every line passing through a green and a yellow point with any other line passing through a green and yellow point lies in the interior of B_2.

Let us note that, from the assumption (1) on α, no line passing through points of the same color intersects B_3. This ends the construction of S'.

We now count the number of different colored radial orderings as seen from all points in B_3. Consider the colored radial ordering seen from a point in the interior of B_3. Note that there is a straight line passing through this point and separating the red and blue points from the green and yellow points. Thus we may assume, that every colored radial ordering as seen from points in the interior of B_3 is written so that all the blue and red points appear before the green and yellow points.

Let p and q be two points in the interior of B_3. Consider the colored radial orderings appearing when walking from p to q in a straight line. By assumption (1), points of the same color will never transpose. Therefore the circular radial (non-colored) ordering of points of the same color will always be the same. Since we are assuming that in the orderings red and blue points are always written first, the k-th red point (or any color for that matter) will always be the same red point. These previous considerations and the fact that we are walking in a straight line, imply that for any k, the number of blue points after the k-th red point is either increasing or decreasing monotonically; the same observation holds for green and yellow points. Therefore in the walk from p to q, once a red-blue (or a yellow-green) swap line is crossed, all colored radial orderings afterwards will be distinct. Thus the number of distinct colored radial orderings of S' as seen from every point in B_3 is equal to the number of cells in the order partition that intersect B_3 in its interior. Since we are assuming that in the interior of B_3 red-blue swap lines only intersect with yellow-green swap lines, simple arithmetic shows that this number is equal to $((m-1)^2+1)^2$. Since $m \geq 8$, this is at least $m^4/2$.

We now proceed to replace the points in S' by patterns of red and blue points, in such a way that the radial orderings as seen from the points in different cells of B_3 remain different. Denote by C_i the cells of the new order partition within B_3 and let p_i be a point inside C_i.

Let $\varepsilon' > 0$. Firstly, every point is placed on its corresponding circle at distance at most ε' of the original position. Recall that these are: the boundary of B_1 for the original blue points; the circle C_1 for the original red points; the boundary of B_2 for the yellow points; and the circle C_2 for the green points. We also assume that all new blue points are above the line passing through the original point and the center of its corresponding circle, while all new red points are below. We choose ε' to be small enough so that the points of every pattern appear consecutively in the radial orderings around each point p_i (this is possible because each point p_i is in the interior of the cell C_i).

Now, the points are replaced by convenient patterns as follows: every blue point with a pattern of one red and one blue point; every red point with a pattern of two red and two blue points; every yellow point with a pattern of three red and three blue points; and every green point with a pattern of four red and four blue points. Note that our choice of patterns implies that two equivalent radial orderings must match patterns of the same type. Also, if necessary, we may assume that α is small enough so that this augmented set is in general position.

We still have to place $20 + r$ points in order to achieve the required number of points. Half of these points are red and half are blue. It is easy to place them so that the final point set is in general position and, in the colored radial ordering given by every point p_i, these final $10 + r/2$ red points appear consecutively as well as the final $10 + r/2$ blue points. This final condition implies that the colored radial orderings as seen from points p_i remain distinct.

Let S be the set of n points resulting from the previous construction. Then S has at least $m^4/2$ colored radial orderings. By letting $\mathbf{c} = \frac{1}{2 \cdot 20^4}$ the result follows. $\square$

We now provide a family of bi-colored point sets with at most $O(n^2)$ colored radial orderings.

Theorem 3.2 *Let n be a multiple of four, then a set S of $n/2$ red and $n/2$ blue points exists with at most $O(n^2)$ colored radial orderings.*

Proof. We employ a technique similar to the one used in the proof of Theorem 3.1. We start with a set S' of $n/4$ points, placed evenly in a circle. All the points of S' have the same color and thus the colored radial orderings of S' are all equivalent. Afterwards, we replace each point of S' with a symmetric pattern of red and blue points. This is done in such a way that the new number of distinct colored radial orderings increases at most to $O(n^2)$.

Let then S' be a set of $n/4$ points placed evenly in the circle $\mathcal{C}$ of radius one and centered at the origin. Explicitly S' consists of the set $\{(\cos(\frac{8\pi}{n} \cdot i), \sin(\frac{8\pi}{n} \cdot i)) \mid i = 1, 2, \ldots, n/4\}$. Let $L(S')$ be the set consisting of all the lines passing through pairs of points in S', together with the lines tangent to $\mathcal{C}$ at each point of S'. It may happen that three lines of $A(S')$ meet at a common point not in S'; this is undesirable for our ulterior purposes. Thus, we assume that S' is actually a set of points $p_1, p_2, \ldots, p_{n/4}$ in $\mathcal{C}$, such that every point p_i is arbitrarily close to the point $(\cos(\frac{8\pi}{n} \cdot i), \sin(\frac{8\pi}{n} \cdot i))$, and such that if three lines in $L(S')$ intersect, they do so at a point in S'. It is not difficult to see that such a set exists. For each point p_i in S' let θ_i be the angle such that $p_i = (\cos(\theta_i), \sin(\theta_i))$.

We now replace each point $p_i = (\cos(\theta_i), \sin(\theta_i))$ in S' with a symmetric pattern of red and blue points as follows. For each $\delta \in (0, 1)$, we define S_δ to be the set of points that results from replacing every point p_i with the following "red,blue,blue,red" pattern: a red point at $(\cos(\theta_i - 2\delta), \sin(\theta_i - 2\delta))$; a blue point at $(\cos(\theta_i - \delta), \sin(\theta_i - \delta))$; a blue point at $(\cos(\theta_i + \delta), \sin(\theta_i + \delta))$; and a red point at $(\cos(\theta_i + 2\delta), \sin(\theta_i + 2\delta))$. Note that this pattern has the property that it looks the same from "above" and "below".

Let $L(S_\delta)$ be the set of lines passing through every pair of points of S_δ. Note that if we let δ tend to 0, then $L(S_\delta)$ tends to $L(S')$. This implies that the line arrangement $A(S_\delta)$ induced by $L(S_\delta)$ tends to the line arrangement $A(S')$ induced by $L(S')$. Note that for some small enough value δ', all line arrangements $A(S_\delta)$ with $\delta < \delta'$ are combinatorially equivalent, i.e., no new cells appear or disappear.

Let us calculate an upper bound on the number of different radial orderings of $S_{\delta'}$ by considering all possible colored radial orderings when seen from a point in the interior of every cell of $A(S')$. Let C be a cell of $A(S_{\delta'})$ and q be a point in the interior of C. There are three different cases according to the limit of C in $A(S')$:

- C tends to a cell of $A(S')$.
  In this case, every pattern replacing a point of S' will appear consecutively in the colored radial ordering around q. Moreover, by the symmetry of the patterns they are all "red,blue,blue,red". Thus in this case there is only one possible colored radial ordering in the cell.

- C tends to an edge of $A(S')$.
  We distinguish two subcases; whether the edge is contained in the line passing through two points p_i and p_j of S' or whether it is contained in a line tangent to the circle $\mathcal{C}$ at a point p_k of S'.

In the first case, the patterns will appear consecutively in the colored radial ordering at points distinct from p_i and p_j. However, around at p_i and p_j, the points in the patterns will appear together but might be intermixed. Since there are only 8 points involved, there is only a constant number of ways in which this can happen. The second case is similar.

- C tends to a vertex of $A(S')$.
  In this case, C may tend to a vertex that is a point of S' or the intersection of two lines ℓ_1 and ℓ_2 of $L(S')$.

  Let us suppose that C tends to a point p_i of S'. Then C must be bounded by lines passing through points of the pattern at p_i and points of S'. Note that in total there are $4n$ lines and thus there are at most $c' \cdot n^2$ such cells, where c' is a positive constant not depending on n. Moreover, since we may assume that S' is arbitrarily close to being evenly placed in the circle $\mathcal{C}$, our analysis does not depend on our choice of p_i. Hence, the cells tending to any other point of S' will induce the same set of colored radial orderings.

  In the second case the lines ℓ_1 and ℓ_2 may be defined by two points or one point of S'. In both cases we have: the points of the patterns at the points of S' defining ℓ_1 appear together but may be intermixed; the points of the patterns at the points of S' defining ℓ_2 appear together but may be intermixed; the patterns at any other points will appear consecutively. Since there is at most of four points defining ℓ_1 and ℓ_2, the number of ways in which their respective pattern points can appear is at most a constant. The only thing left to consider is the $O(n)$ number of ways in which the points of the patterns at the points defining ℓ_1 can appear with respect to those of ℓ_2.

Note that the total number of distinct colored radial orderings of $S_{\delta'}$ is at most $O(n^2)$. Therefore by setting $S := S_{\delta'}$, the result follows.

$\square$

We now give a linear lower bound for the number of colored radial orderings. Some notation is required. For a given radial ordering σ of S, let $\sigma(i)$ be its $(i+1)$-element. Therefore $\sigma = [\sigma(0), \sigma(1), \ldots, \sigma(n-1)]$. Two radial orderings σ and ρ are equivalent whenever there exists a natural number j such that $\sigma(i) = \rho(i+j)$ for all i (where addition is taken modulo n); they are equivalent as colored radial orderings when the color of $\sigma(i)$ is equal to the color of $\rho(i+j)$. We now provide a lower bound of $n/2$ on the number of distinct colored radial orderings of S.

Theorem 3.3 *The number of distinct colored radial orderings of S is at least $n/2$.*

Proof. Through out the proof we use both *colored* and *un-colored* radial orderings. In each instance we explicitly mention to which of the two types of radial orderings we are referring to.

To obtain the claimed lower bound, we show a walk in which $n/2$ distinct *colored* radial orderings are seen. First choose a red point p of S and let $\mathcal{C}$ be a circle centered at p. Afterwards walk once clockwise around $\mathcal{C}$. Choose $\mathcal{C}$ to be small enough so that the only swap lines of S crossed in the walk are those involving p.

Consider the *un-colored* radial orderings seen in this walk. Note that since we do not cross any swap line defined by points of $S \setminus \{p\}$, the points of $S \setminus \{p\}$ remain fixed in these radial orderings. The only point that changes position is p; it moves counter clockwise, by transposing an element of $S \setminus \{p\}$ every time a swap line is crossed. We prove that every time that p transposes a blue point, a new *colored* radial ordering is seen. This implies a lower bound of $n/2$ on the number of *colored* radial orderings seen in the walk.

Since in the walk around $\mathcal{C}$, the subset of blue points of S always appear in the same *un-colored* radial ordering, we assume that all the *un-colored* radial orderings of S as seen from points in $\mathcal{C}$ are written starting at the same blue point.

Among these radial orderings, for each $k = 0, \dots n/2 - 1$, let σ_k be the radial ordering in which p is just after the k-th blue element. We show that all the *colored* radial orderings associated to these $n/2$ *un-colored* radial orderings are distinct.

Let then σ_k and σ_l be two such *un-colored* radial orderings encountered at points q_1 and q_2 of $\mathcal{C}$ respectively. Suppose that σ_k and σ_l are equivalent as *colored* radial orderings. Then there exist a fixed natural number j such that for all $i = 0, \dots, n-1$, the color of $\sigma_k(i)$ is equal to the color of $\sigma_l(i+j)$.

We now define a graph that captures the relationship between σ_k and σ_l; we employ the structure of this graph to conclude that k must equal l. Let then, G be the directed graph whose vertex set is the set of points S and in which, for all $i = 0, \dots n-1$ there is an arc from $\sigma_k(i)$ to $\sigma_l(i+j)$. Note that every vertex in G has indegree and outdegree equal to 1. Therefore G is the union of disjoint directed cycles and each cycle consists of points of the same color.

Let Γ be the cycle containing p and let $S' := S \setminus V(\Gamma)$. Let ρ_1 and ρ_2 be the *un-colored* radial orderings of S' as seen from q_1 and q_2 respectively. We make the additional assumption that ρ_1 is written starting at $\sigma_k(0)$ while ρ_2 is written starting at $\sigma_l(n-j)$ (note that these points are not in Γ). Since in particular $p \notin S'$, these radial orderings are equivalent. Therefore there exists a fixed natural number j' such that $\rho_1(i) = \rho_2(i+j')$, for all i. Note also that since S' comes from removing from S the points of a cycle of G, the color of $\rho_1(i)$ is equal to the color of $\rho_2(i)$.

Let G' be the directed graph whose vertex set is S' and in which there is an arc from $\rho_1(i)$ to $\rho_2(i)$. As before every edge in G' has indegree and outdegree equal to 1. Therefore G' is the union of disjoint cycles (in fact G' is the subgraph of G induced by S'). Since ρ_2 is just a "shift" of j' places to the right of ρ_1, all of these cycles have the same length, let m be this length. Therefore, both the number of red and blue points in S' are multiples of m. This implies that the number of vertices in Γ is also a multiple of m.

Let $r \cdot m$ be the length of Γ, since Γ is non empty, then $r \geq 1$. Assume that Γ, starting from p is given by $(p = v_1, v_2, \dots, v_m, \dots, v_{2m}, \dots, v_{rm})$. Let b_k be the k-th blue point and $\Gamma' = (b_k = u_1, \dots, u_m)$ be the cycle in G containing b_k. Consider the following sequence of pairs of vertices $(u_1, v_1), (u_2, v_2), \dots, (u_m, v_m)$. Note that in σ_k, the point v_1 is just after the point u_1 and there after for $2 \leq i \leq m$, the point v_i is just after the point u_i in both σ_l and σ_k. Suppose that $r > 1$, then the point v_{m+1} is just after u_1 in σ_l while in σ_k it is just after the point v_1(which is equal to p). We repeat the previous observations, but for the following sequence of vertices $(v_2, v_{m+2}), (v_3, v_{m+3}), \dots, (v_{(r-1)m+1}, v_{rm+1} = p)$ and note that for $m+2 \leq i \leq rm$ the point v_i is just after the point v_{i-m} in both σ_k and σ_l. For $i = rm+1$, the point v_{rm+1} is just after the point $v_{(r-1)m+1}$ in σ_l, but $v_{rm+1} = p$ and $v_{(r-1)m+1}$ is red, a contradiction since p is just after a blue point in σ_l. Thus $r = 1$. This implies that $k = l$ and the result follows. $\square$

We provided examples of bi-colored sets of n points (half of them red and half of them blue) with at most $O(n^2)$ and at least $\Omega(n^4)$ colored radial orderings. We finalize this section by noting that every intermediate value is achievable. Let S_1 and S_2 be the two previous examples. Consider a biyective function $f : S_1 \to S_2$, that takes red points to red points and blue points to blue points. Move the points of S_1 continuously one by one to their image under f. In the process take note of the changes in the order partition. The order partition changes when three points become collinear and a cell collapses to a point. Thereafter a new cell is created. The colored radial ordering induced in the new and previous cell might be different but the colored radial orderings induced by the other cells stay the same. This observation implies that the number of distinct colored radial orderings at each of these steps changes at most by one. Therefore in this continuous motion every number of different colored radial orderings between those of S_1 and those of S_2 is achieved.

4 Conclusions

We proved an upper bound of $O(n^4)$ and a lower bound of $\Omega(n^2)$ on the number of radial orderings every set of n points in the plane must have. We leave the closing of this gap an open problem.

Conjeture 4.1 *The number of distinct radial orderings of a set of n points in general position in the plane is $\Theta(n^4)$.*

For the colored case we gave an example with $\Omega(n^2)$ colored radial orderings and proved that every bi-colored set of points (with equal sized color classes) has at least $n/2$ distinct radial orderings.

Conjeture 4.2 *For n even, the number of distinct colored radial orderings of a set of $n/2$ red points and $n/2$ blue points in general position in the plane is at least $\Omega(n^2)$.*

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