Locating a service facility and a rapid transit line

J. M. Díaz-Báñez* M. Korman† P. Pérez-Lantero‡ I. Ventura*

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Abstract

In this paper we study a facility location problem in the plane in which a single point (facility) and a rapid transit line (highway) are simultaneously located in order to minimize the total travel time of the clients to the facility, using the L_1 or Manhattan metric. The rapid transit line is represented by a line segment with fixed length and arbitrary orientation. The highway is an alternative transportation system that can be used by the clients to reduce their travel time to the facility. This problem was introduced by Espejo and Chía in [7]. They gave both a characterization of the optimal solutions and an algorithm running in $O(n^3 \log n)$ time, where n represents the number of clients. In this paper we show that the Espejo and Chía's algorithm does not always work correctly. At the same time, we provide a proper characterization of the solutions with a simpler proof and give an algorithm solving the problem in $O(n^3)$ time.

Keywords: Geometric optimization; Facility location; Transportation; Time distance.

1 Introduction

Suppose that we have a set of clients represented as a set of points in the plane, and a service facility represented as a point to which all clients have to move. Every client can reach the facility directly or by using an alternative rapid transit line or highway, represented by a straight line segment of fixed length and arbitrary orientation, in order to reduce the travel time. Whenever a client moves directly to the facility, it moves at unit speed and the distance traveled is the Manhattan or L_1 distance to the facility. In the case where a client uses the highway, it travels the L_1 distance at unit speed to one endpoint of the highway, traverses the entire highway with a speed greater than one, and finally travels the L_1 distance from the other endpoint to the facility at unit speed. All clients traverse the highway at the same speed. Given the set of points representing the clients, the facility location problem consists in determining at the same time the facility point and the highway in order to minimize the *total weighted travel time* from the clients to the facility. The weighted travel time of a client is its travel time multiplied by a weight representing the intensity of its demand. This problem was studied by Espejo and Chía [7]. We refer to [7] and references therein to review both the state of the art and applications of this problem.

Geometric problems related to transportation networks have been recently considered in computational geometry. Abellanas *et. al.* introduced the *time metric* model in [1]: Given an underlying

*Departamento de Matemática Aplicada II, Universidad de Sevilla, Spain. Partially supported by project MEC MTM2009-08652. {dbanez,iventura}@us.es.

†Université libre de Bruxelles (ULB). mkormanc@ulb.ac.be.

‡Departamento de Computación, Universidad de Valparaíso, Chile. Partially supported by project MEC MTM2009-08652. pablo.perez@uv.cl

metric, the user can travel at speed v when moving along a highway h or unit speed elsewhere. The particular case in which the underlying metric is the L_1 metric and all highways are axis-parallel segments of the same speed, is called the *city metric* [3]. The optimal positioning of transportation systems that minimize the maximum travel time among a set of points has been investigated in detail in recent papers [2, 6, 4]. Other more general models are studied in [8]. The variant introduced by Espejo and Chía aims to minimize the sum of the travel times (transportation cost) from the demand points to the new facility service, which has to be located simultaneously with a highway. The highway is used by a demand point whenever it saves time to reach the facility.

Notation to formulate the problem is as follows. Let S be the set of n client points; f the service facility point; h the highway; ℓ the length of h; t and t' the endpoints of h; and $v > 1$ the speed in which the points move along h. Let $w_p > 0$ be the weight (or demand) of a client point p. Given a point u of the plane, let $x(u)$ and $y(u)$ denote the x- and $y-$coordinates of u respectively. The distance or travel time (see Figure 1), between a point p and the service facility f is given by the function

$$d_{t,t'}(p, f) := \min\{\|p - f\|_1, \|p - t\|_1 + \frac{\ell}{v} + \|t' - f\|_1, \|p - t'\|_1 + \frac{\ell}{v} + \|t - f\|_1\}.$$

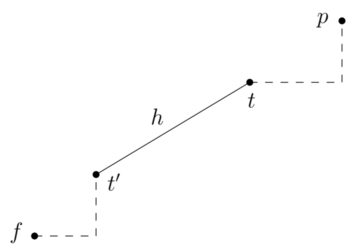

Figure 1: The distance between a point p and the facility f using the highway.

Then the problem can be formulated as follows:

> **The Facility and Highway Location problem (FHL-problem)**: Given a set S of n points, a weight $w_p > 0$ associated with each point p of S, a fixed highway length $\ell > 0$, and a fixed speed $v > 1$, locate a point (facility) f and a line segment (highway) h of length ℓ with endpoints t and t' such that the function $\sum_{p \in S} w_p \cdot d_{t,t'}(p, f)$ is minimized.

Espejo and Chía [7] studied the FHL-problem and gave the following characterization of the solutions. Consider the grid G defined by the set of all axis-parallel lines passing through the elements of S. They stated that there always exists an optimal highway having one endpoint at a vertex of G. Based on this, they proposed an $O(n^3 \log n)$-time algorithm to solve the problem. In this paper we show that the characterization given by Espejo and Chía is not true, hence their algorithm does not always give the optimal solution.

In Section 2 we first provide a proper characterization of the solutions. Our proof uses geometric observations and is simpler than the proof given in [7]. After that we give a counterexample to the Espejo and Chía's characterization. We provide a set of five points, all having weight equal to one, and prove that no optimal highway has one endpoint in a vertex of G. In Section 3 we present an improved algorithm running in $O(n^3)$ time that correctly solves the FHL-problem. Finally, in Section 4 we state our conclusions and proposal for further research.

2 Properties of an optimal solution

A primary observation (also stated in [7]) is that the service facility can be located at one of the endpoints of the rapid transit line. From now on, we assume throughout the paper that $f = t'$. This assumption simplifies the distance from a point $p \in S$ to the facility to the following expression,

$$d_t(p, f) = \min\{\|p - f\|_1, \|p - t\|_1 + \frac{\ell}{v}\}.$$

We say that a point p uses the highway if $\|p - t\|_1 + \frac{\ell}{v} < \|p - f\|_1$, and that p does not use it (or goes directly to the facility) otherwise. Given f and t, we say that the bisector of f and t is the set of points z such that $\|z - f\|_1 = \|z - t\|_1 + \frac{\ell}{v}$, see Figure 2. A geometrical description of such bisector can be found in [7].

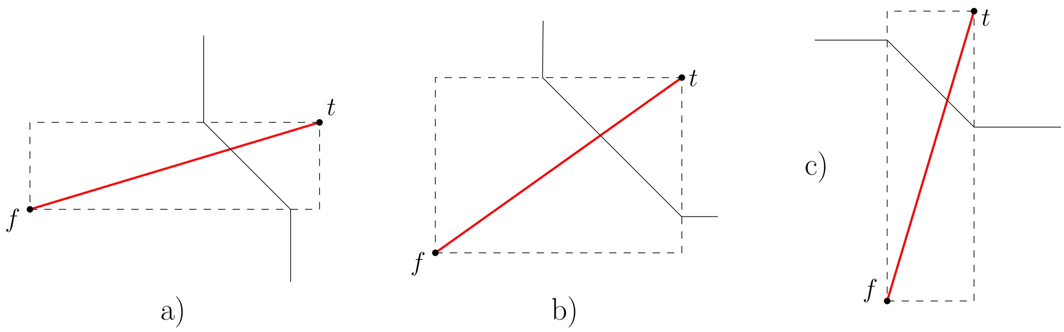

Figure 2: The bisector of f and t.

Lemma 2.1 *There exists an optimal solution to the FHL-problem satisfying one of the next conditions:*

- *(a) One of the endpoints of the highway is a vertex of G.*
- *(b) One endpoint of the highway is on a horizontal line of G, and the other endpoint is on a vertical line of G.*

Proof. Let f and t be the endpoints of an optimal highway h and assume that neither of the two conditions is satisfied. Using local perturbation we will transform this solution into one that satisfies one of the two conditions. Assume that neither f nor t is on vertical lines of G. Let $\delta_1 > 0$ (resp. $\delta_2 > 0$) be the smallest value such that if we translate h with vector $(-\delta_1, 0)$ (resp. $(\delta_2, 0)$) then one endpoint of h touches a vertical line of G. Given $\varepsilon \in [-\delta_1, \delta_2]$, let f_ε, t_ε, and h_ε be f, t, and h translated with vector $(\varepsilon, 0)$, respectively. It is easy to see that $|d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)| = \varepsilon$ for all points p. Given a real number x, let $\mathrm{sgn}(x)$ denote the sign of x. We partition S into three sets S_1, S_2 and S_3 as follows:

$$\begin{aligned} S_1 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = \mathrm{sgn}(\varepsilon), \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\} \\ S_2 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = -\,\mathrm{sgn}(\varepsilon), \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\} \\ S_3 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = -1, \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\} \end{aligned}$$

Note that the elements of S_3 belong to the bisector of f and t, S_1 contains the demand points that travel leftwards to reach f, and S_2 contains the points that travel rightwards.

By the linearity of the L_1 metric, whenever we translate the highway ε units to the right (for some arbitrarily small ε, $0 < \varepsilon \leq \delta_1$), the highway will be ε units closer for points in $S_2 \cup S_3$, but ε units further away for points of S_1. Analogously, the distance to the facility decreases for points in $S_1 \cup S_3$ and increases for points of S_2 when translating h leftwards. Given $X \subseteq S$, let $w(X) = \sum_{p \in X} w_p$ be the sum of weights of the points in set X. Let $N = w(S_1) - w(S_2)$ and $k = w(S_3)$. Thus, for any vector $(\varepsilon, 0)$, $\varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}$, the change of the objective function when we traslate the highway with vector $(\varepsilon, 0)$ is as follows,

$$\sum_{p \in S} w_p \cdot d_{t_\varepsilon}(p, f_\varepsilon) - \sum_{p \in S} w_p \cdot d_t(p, f) = w(S_1)\varepsilon - w(S_2)\varepsilon - w(S_3)|\varepsilon| = N\varepsilon - k|\varepsilon|.$$

We claim that $k = 0$ and $N = 0$. In fact, since h is optimal $N\varepsilon - k|\varepsilon| \geq 0$ for all $\varepsilon \in [-\delta_1, \delta_2]$. If $\varepsilon > 0$ then $N \geq k$. Otherwise, $\varepsilon < 0$ implies $N \leq -k$. Therefore, if $k > 0$ then we have a contradiction, that is, $N \leq -k$ and $N \geq k$. Thus $k = 0$ implying $N = 0$. Therefore we can translate h either rightwards or leftwards in such a way the objective function keeps unchanged. This operation can be done until f or t is on a vertical line of G. We repeat the same operation on the y coordinates and also obtain that one of the two endpoints must be on a horizontal line of G, hence satisfying one of the two conditions of the Lemma. □

In [7] the authors stated that an optimal solution always exists satisfying Lemma 2.1 (a). Unfortunately, the above claim is false and their algorithm may miss some highway locations; indeed, it may miss the optimal location and thus fail. We provide here one counterexample and the following result.

Lemma 2.2 *There exists a set of unweighted points in which no optimal solution to the FHL-problem satisfies Lemma 2.1 (a).*

Proof. Consider five points with coordinates $(-4, 0)$, $(-3, -1)$, $(12, 8)$, $(13, 5)$, and $(13, 7)$, all having weights equal to one. See Figure 3. Let $\ell = \sqrt{180}$ be the length of the highway h and $v > 1$ be the speed. We claim that the highway with endpoints $f = (0, 0)$ and $t = (12, 6)$ is better than any other solution with an endpoint at a vertex of G.

If one endpoint of h is a vertex of G in the line $x = -3$, then the other endpoint is located to the left of the line $x = 12$ because $\ell = \sqrt{180} < 12 - (-3) = 15$. In that case we can translate h rightwards with vector $(\frac{1}{2}, 0)$ improving the objective function. The same holds if one endpoint of h is a vertex in the line $x = -4$. Similarly, if one endpoint is a vertex in the line $x = 13$ then we can translate h leftwards with vector $(-\frac{1}{2}, 0)$ and the objective function decreases. Additionally, if one endpoint has coordinates $(12, 0)$ or $(12, -1)$ then it is straighforward to see that h can be translated or rotated in order to decrease the objective function.

Consider now the cases in which one of the endpoints has coordinates $(12, y_0)$ for some $y_0 \in \{5, 7, 8\}$. We start by showing that, in any of the three cases, the optimal position of the other endpoint of the highway (denoted by e) must lie on the line $y = 0$. Since the highway's length is fixed, the possible positions of e lie in an (Euclidean) arc or radius ℓ. Observe that the clients that walk to e are points $a = (-4, 0)$ and $b = (-3, 1)$. Also notice that for any $y_0 \in \{5, 7, 8\}$ we have $x(e) > \max\{x(a), x(b)\}$ (see Figure 4 a)). Hence, we are interested in minimizing the expression:

$$||a - e||_1 + ||b - e||_1 = 2x(e) + cy(e) - x(a) - y(a) - x(b) - y(b) = 2x(e) + cy(e) + 6$$

Where $c = 2$ (if $y(e) \geq 0$), $c = 0$ (if $-1 \geq y \geq 0$) or $c = -2$ (otherwise). It is easy to see that this expression is minimized when $y(e) = 0$ (see Figure 4 b)), hence whenever one of the

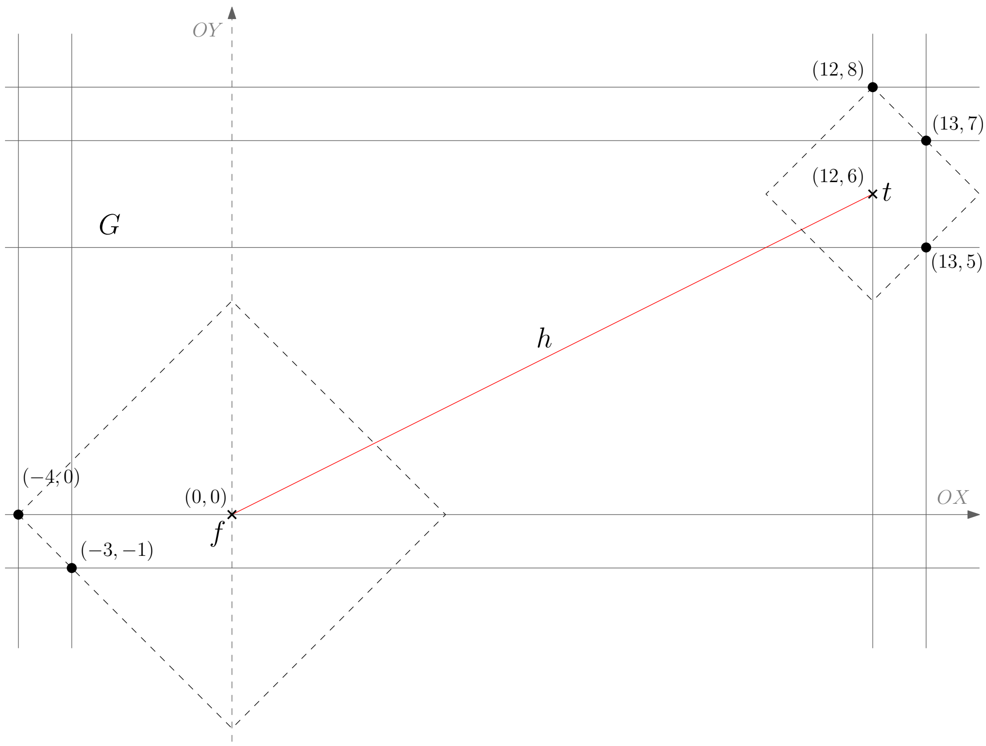

Figure 3: A counterexample to the algorithm of Espejo and Chía.

highway endpoints has coordinates $(12, y_0)$, for some $y_0 \in \{5, 7, 8\}$, the other endpoint e must satisfy $y(e) = 0$.

If $y_0 = 8$, then h can be translated downwards with vector $(0, -\frac{1}{2})$ and the objective function decreases. Thus point $(12, 8)$ is discarded. It remains to show that there is a solution better than the one having an endpoint at either $(12, 7)$ or $(12, 5)$, and the other endpoint on the line $y = 0$. It is as follows.

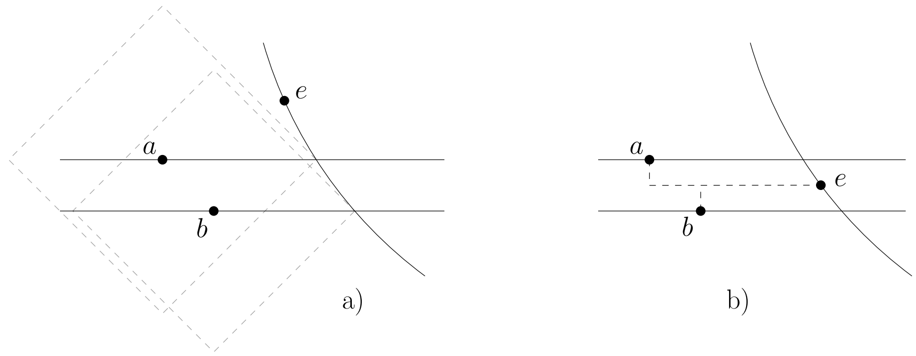

Figure 4: $a = (-4, 0)$ and $b = (-3, -1)$. When one endpoint of the highway has coordinates $(12, 8)$, $(12, 7)$, or $(12, 5)$, the optimal position of the other endpoint e is on the line $y = 0$.

Given a value ε, let f_ε be the point with coordinates $(\varepsilon, 0)$ and t_ε be the point in the line $x = 12$ such that $y(t_\varepsilon) > 0$ and the Euclidean distance between f_ε and t_ε is equal to ℓ (see Figure 5). Let $[-\delta_1, \delta_2]$, $\delta_1, \delta_2 > 0$, be the maximal-length interval such that $5 \leq y(t_\varepsilon) \leq 7$ for all $\varepsilon \in [-\delta_1, \delta_2]$.

The variation of the objective function when f and t are moved to f_ε and t_ε, respectively, is

$$g(\varepsilon) := 2|x(f_\varepsilon) - x(f)| - |y(t_\varepsilon) - y(t)| = 2\varepsilon - \left(\sqrt{36 + 24\varepsilon - \varepsilon^2} - 6\right).$$

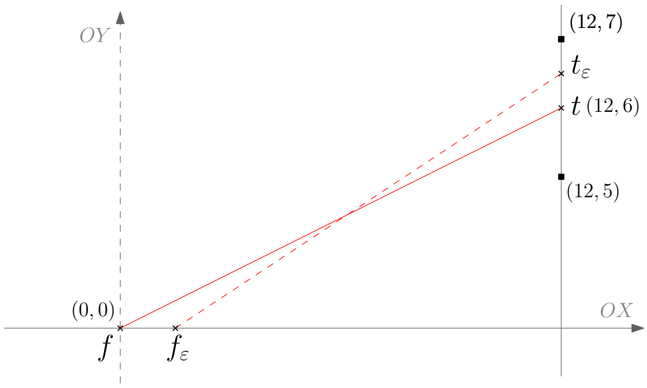

Figure 5: Definitions of f_ε and t_ε.

It can be easily proved using analytic arguments that $g(\varepsilon) > 0$ for all $\varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}$. Then the highway with endpoints f and t gives a better solution than that having an endpoint at $(12, 7)$ or $(12, 5)$. This completes the proof.

□

In the next section we provide a correct algorithm that solves the problem in $O(n^3)$ time. We assume general position, that is, there are no two points on a same line having slope in the set $\{-1, 0, 1, \infty\}$.

3 The algorithm

Lemma 2.1 can be used to find an optimal solution to the FHL-problem. Although the method is quite similar for both cases in Lemma 2.1, we address the two cases independently for the sake of clarity. By Vertex-FHL-problem we will denote the FHL-problem for the cases in which Lemma 2.1 a) holds, and by Edge-FHL-problem the FHL-problem for the cases in which Lemma 2.1 b) holds. In the next subsections we give an $O(n^3)$-time algorithm for each variant of the problem. In both of them we assume w.l.o.g. that the highway length ℓ is equal to one.

In the following θ will denote the positive angle of the highway with respect to the positive direction of the x-axis. For the sake of clarity, we will assume that $\theta \in [0, \frac{\pi}{4}]$. When θ belongs to the interval $[k\frac{\pi}{4}, (k+1)\frac{\pi}{4}]$, $k = 1, \ldots, 7$, both the Vertex- and Edge-FHL-problem can be solved in a similar way.

Given a point u and an angle θ, let $u(\theta)$ be the point with coordinates $(x(u) + \cos\theta, y(u) + \sin\theta)$. There exists an angle $\phi \in [0, \frac{\pi}{4}]$ such that the bisector of the endpoints f and $t = f(\theta)$ has the shape in Figure 2 a) for all $\theta \in [0, \phi)$, and has the shape in Figure 2 b) for all $\theta \in (\phi, \frac{\pi}{4}]$. Such an angle ϕ verifies $\cos(\phi) - \sin(\phi) = \frac{1}{v}$. Furthermore, $\phi = \frac{1}{2}\arcsin(1 - \frac{1}{v^2})$ and $\phi \neq \frac{\pi}{4}$ unless v is infinity.

Let Π_x, Π_y, and Π_{x+y} denote the point set S sorted according to the x-, y-, and $(x + y)$-order, respectively.

3.1 Solving the Vertex-FHL-problem

For each vertex u of G we can solve the problem subject to $f = u$ or $t = u$. We show how to obtain a solution if $f = u$. The case where $t = u$ can be solved analogously.

Suppose w.l.o.g. that the vertex $f = u$ is the origin of the coordinate system and the highway angle is θ, for $\theta \in [0, \frac{\pi}{4}]$. Thus, the distance between a point $p \in S$ and the facility u has the expression $c_1 + c_2 \cos\theta + c_3 \sin\theta$, where $c_1 > 0$ and either $c_2, c_3 = \pm w_p$ (p uses the highway) or $c_2 = c_3 = 0$ (p does not use the highway). When θ goes from 0 to $\frac{\pi}{4}$ this expression changes at the values of θ such that:

- The point p switches from using the highway to going directly to the facility (or vice versa). We call these changes *bisector events*. A bisector event occurs when the bisector between the highway's endpoints u and $u(\theta)$, contains p. At most two bisector events are obtained for each point p.
- The highway endpoint $u(\theta)$ crosses the vertical or horizontal line passing through p. We call this event *grid event*. Again, each point of S generates at most two grid events.
- $\theta = \phi$. This event is called *ϕ-event*.

Lemma 3.1 *After an $O(n \log n)$-time preprocessing, the angular order of all the events associated with a given vertex of G can be obtained in linear time.*

Proof. The preprocessing consists in computing Π_x, Π_y, and Π_{x+y}, which can be done in $O(n \log n)$ time. Now, let u be a vertex of G. It is straightforward to see that they are $O(n)$ grid events and that we can obtain their angular order in linear time by using both Π_x and Π_y. Let us show how to obtain the bisector events in $O(n)$ time.

The bisector of u and $u(\theta)$ consists of two axis-aligned half-lines and a line segment with slope -1 connecting their endpoints (see Figure 2 and [7] for further details). Given a point p, when θ goes from 0 to $\pi/4$ the bisector between u and $u(\theta)$ passes through p at most twice, that is, when p belongs to one of the half-lines of the bisector and when p belongs to the line segment. If p belongs to the line segment of the bisector then the event is denoted by α_p (see Figure 6 b)). If p belongs to the leftmost half-line of the bisector, which is always vertical, we denote that event by β_p (see Figure 6 a)). Otherwise, if p belongs to the rightmost half-line which can be either vertical or horizontal we denote that event by γ_p (see Figure 6 c) and d)). Observe that if the rightmost half-line is vertical then $\gamma_p < \phi$, otherwise $\gamma_p > \phi$.

Let Π_1 be the subsequence of Π_{x+y} containing all elements p such that $\alpha_p \in [0, \frac{\pi}{4}]$, Π_2 be the subsequence of Π_x containing all elements p such that $\beta_p \in [0, \frac{\pi}{4}]$, and Π_3 be the subsequence of Π_x that contains all elements p such that $y(p) < y(u)$ and $\gamma_p \in [0, \frac{\pi}{4}]$, concatenated with the subsequence of Π_y that contains all elements p such that $x(p) > x(u)$ and $\gamma_p \in [0, \frac{\pi}{4}]$. Given a point $p \in S$, the corresponding events of p in $[0, \frac{\pi}{4}]$ can be found in constant time, thus Π_1, Π_2, and Π_3 can be built in linear time.

The following statements are true for any point $p \in S$:

(a) $x(p) + y(p) = \frac{1}{2}(\cos\alpha_p + \sin\alpha_p + \frac{1}{v})$ for all points p in Π_1.

(b) $x(p) = \frac{1}{2}(\cos\beta_p - \sin\beta_p + \frac{1}{v})$ for all points p in Π_2.

(c) $x(p) = \frac{1}{2}(\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p < \phi$.

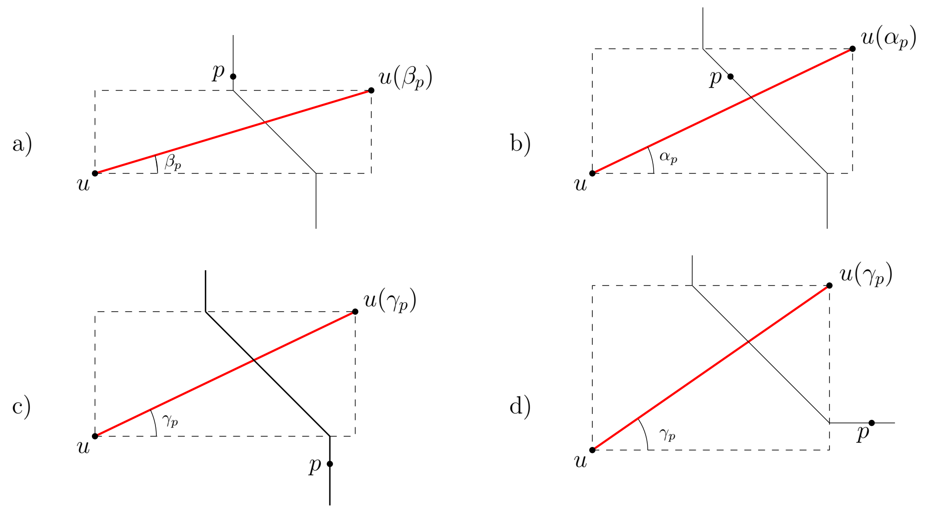

Figure 6: The bisector events of p when $\theta \in [0, \frac{\pi}{4}]$. a) p belongs to the leftmost half-line of the bisector of u and $u(\theta)$. b) p belongs to the segment. c,d) p belongs to the rightmost half-line of the bisector.

(*d*) $y(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p > \phi$.

Let Γ_1 (resp. Γ_2, Γ_3) be the sequence obtained by replacing each element p in Π_1 (resp. Π_2, Π_3) by α_p (resp. β_p, γ_p). Therefore, from statements $(a) - (d)$ and the monotonicity of the functions $\cos\theta + \sin\theta$, $\cos\theta - \sin\theta$, and $-\cos\theta + \sin\theta$ in the interval $[0, \frac{\pi}{4}]$, we obtain that Γ_1, Γ_2, and Γ_3 are sorted sequences. Using a standard method for merging sorted lists, we can merge in linear time Γ_1, Γ_2, Γ_3, the grid events, and the ϕ-event. Therefore, the angular order of all events associated with a vertex u can be obtained in $O(n)$ time and the result follows. □

Theorem 3.2 *The Vertex-FHL-problem can be solved in $O(n^3)$ time.*

Proof. Let u be a vertex of G. Using Lemma 3.1, we obtain in linear time the angular order of the $O(n)$ events associated with u. The events induce a partition of $[0, \frac{\pi}{4}]$ into maximal intervals. For each of those intervals, the objective function takes the form $g(\theta) := \sum_{p \in S} w_p \cdot d_t(p, u) = b_1 + b_2 \cos\theta + b_3 \sin\theta$. This problem is of constant size in each subinterval and the minimum of $g(\theta)$ can be found in $O(1)$ time. Furthermore, the expression of $g(\theta)$ can be updated in constant time when θ crosses an event point distinct of ϕ when going from 0 to $\frac{\pi}{4}$. In the case where θ crosses ϕ, $g(\theta)$ can be updated in at most $O(n)$ time. Then the problem subject to $f = u$ can be solved in linear time. The case in which $t = u$ can be addressed in a similar way. It gives an overall $O(n^3)$ time complexity because G has $O(n^2)$ vertices. □

3.2 Solving the Edge-FHL-problem

We use now Lemma 2.1 b) to find an optimal solution. Namely, for each horizontal line e_h of G and each vertical line e_v of G, we solve the problem subject to $f \in e_h$ and $t \in e_v$. The reverse can be solved analogously. The solution of the Edge-FHL-problem is similar to the solution of the Vertex-FHL-problem, thus we will skim over some details.

Suppose w.l.o.g. that e_h and e_v intersect at the origin o of the coordinate system. Let $\theta \in [0, \frac{\pi}{4}]$ be the positive angle of the highway with respect to the positive direction of the x-axis and $f = x_\theta$, $t = y_\theta$ be the highway endpoints. See Figure 7.

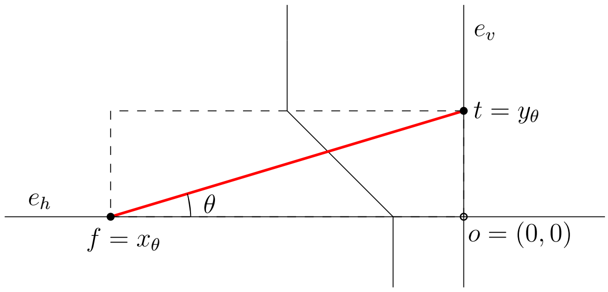

Figure 7: Solving the Edge-FHL-problem.

As in the Vertex-FHL-problem, the distance between p and the facility f can be expressed as $c_1 + c_2 \cos\theta + c_3 \cos\theta$ for some $c_1 > 0$ and $c_2, c_3 \in \{1, -1\}$. When θ goes from 0 to $\frac{\pi}{4}$, f and t move on e_h and e_v respectively and the distance formula changes at the same events stated in the Vertex-FHL-problem: bisector-, grid- and ϕ- events.

Lemma 3.3 *After an $O(n \log n)$-time preprocessing, the angular order of all the events associated with a pair of perpendicular lines of G can be obtained in linear time.*

Proof. We can follow the arguments of Lemma 3.1. Firstly, we note that there are $O(n)$ grid events and their angular order can be obtained in linear time by using both Π_x and Π_y.

Given a point $p \in S$, let the events α_p, β_p, and γ_p be defined as in the Vertex-FHL case. Refer to Figure 6. Let Π_1 be the subsequence of Π_{x+y} containing all elements p such that $\alpha_p \in [0, \frac{\pi}{4}]$, Π_2 be the subsequence of Π_x containing all elements p such that $\beta_p \in [0, \frac{\pi}{4}]$, and Π_3 be the subsequence of Π_x that contains all elements p such that $y(p) < y(o)$ and $\gamma_p \in [0, \frac{\pi}{4}]$, concatenated with the subsequence of Π_y that contains all elements p such that $x(p) > x(o)$ and $\gamma_p \in [0, \frac{\pi}{4}]$. Note that Π_1, Π_2, and Π_3 can be built in linear time.

Given a point $p \in S$, the following statements are true:

(a) $x(p) + y(p) = \frac{1}{2}(-\cos\alpha_p + \sin\alpha_p + \frac{1}{v})$ for all points p in Π_1.

(b) $x(p) = \frac{1}{2}(-\cos\beta_p - \sin\beta_p + \frac{1}{v})$ for all points p in Π_2.

(c) $x(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p < \phi$.

(d) $y(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p > \phi$.

Let Γ_1 (resp. Γ_2, Γ_3) be the sequence obtained by replacing each element p in Π_1 (resp. Π_2, Π_3) by α_p (resp. β_p, γ_b). Therefore, by using similar arguments to those used in Lemma 3.1 the angular order of all events can be obtained in $O(n)$ time, after the lists Π_x, Π_y and Π_{x+y} have been precomputed. □

Theorem 3.4 *The Edge-FHL-problem can be solved in $O(n^3)$ time.*

Proof. Proof is identical to the proof of Theorem 3.2 and thus omitted. □

4 Concluding remarks

In this paper we have addressed a facility location problem on the plane in which a single service facility and a rapid transit line are simultaneously located in order to minimize the total travel time of the clients to the facility, under the L_1 metric. The same problem was considered previously in [7]. We have shown that the characterization of the solutions given in [7] is not true in general and we have provided a proper characterization. Furthermore, we have offered an algorithm that solves the problem in $O(n^3)$ time.

As further research, it would be worth studying the same problem in other metrics or using different optimization criteria. Another interesting variant would be to consider the problem when the length of the highway is not given in advance and it is a variable in the problem. Additionally, we could consider a similar distance model in which the clients can enter and exit the highway at any point (called *freeway* in [5]) .

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