The 1-Median and 1-Highway problem

J. M. Díaz-Báñez[*]   M. Korman[†]   P. Pérez-Lantero[‡]   I. Ventura[*]

September 12, 2012

Abstract

In this paper we study a facility location problem in the plane in which a single point (median) and a rapid transit line (highway) are simultaneously located in order to minimize the total travel time of the clients to the facility, using the L_1 or Manhattan metric. The highway is an alternative transportation system that can be used by the clients to reduce their travel time to the facility. We represent the highway by a line segment with fixed length and arbitrary orientation. This problem was introduced in [Computers & Operations Research 38(2) (2011) 525-538]. They gave both a characterization of the optimal solutions and an algorithm running in $O(n^3 \log n)$ time, where n represents the number of clients. In this paper we show that the previous characterization does not work in general. Moreover, we provide a complete characterization of the solutions and give an algorithm solving the problem in $O(n^3)$ time.

Keywords: Location; Geometric optimization; Transportation; Time distance.

1 Introduction

Suppose that we have a set of clients represented as a set of points in the plane, and a service facility represented as a point to which all clients have to move. Every client can reach the facility directly or by using an alternative highway or rapid transit line, represented by a straight line segment of fixed length and arbitrary orientation, in order to reduce the travel time. Whenever a client moves directly to the facility, it moves at unit speed and the distance traveled is measured with the Manhattan distance to the facility. In the case where a client uses the highway, it travels the L_1 distance at unit speed to an endpoint of the highway, traverses the entire highway using the euclidean distance with a speed $v > 1$, and finally travels the L_1 distance from the other endpoint to the facility at unit speed. All clients traverse the highway at the same speed, and use the highway only whenever it results in a smaller travel time. Given the set of points representing the clients, the facility location problem consists in determining at the same time the facility point and the highway in order to minimize the *total weighted travel time* from the clients to the facility. The

weighted travel time of a client is its travel time multiplied by a weight representing the intensity of its demand.

The above problem can be seen as one urban planning one for which, in addition to a highway, one must locate additional features (in this case a facility). Recently, there has been a renewed interest in shortest path computation under the presence of highways, and highway location-related problems. Abellanas *et al.* [1] introduced the *time distance* model in which, given an underlying metric, the user can travel at speed $v > 1$ when moving along a highway h or unit speed elsewhere. The particular case in which the underlying metric is the L_1 metric and all highways are axis-parallel segments was studied by Aichholzer *et al.* [3]. The optimal positioning of transportation systems that minimize the maximum travel time between a set of points has been investigated in detail in recent papers for both the L_1 metric [2], and the Euclidean metric [5]. A more systematic study of highway location under the presence of other highways and/or obstacles has been done in [14, 13].

The particular problem we study was introduced by Espejo and Rodríguez-Chía [10]. They studied the problem and characterized optimal solutions for any problem instance. Based on their characterization, an $O(n^3 \log n)$-time algorithm to solve the problem was also given. Unfortunately, in a preliminary version of this paper [7, 8] we showed that their characterization is wrong, hence the algorithm of Espejo and Rodríguez-Chía does not always give the optimal solution (this fact was afterwards acknowledged in a corrigendum by the authors [11]). In this paper, we provide a correct characterization of the solution. Moreover, we provide an algorithm that solves the problem, and runs in $O(n^3)$ time.

1.1 Definitions

We formulate the problem as follows. Let S be the set of n client points; f the service facility point; h the highway; ℓ the length of h; t and t' the endpoints of h; and $v \geq 1$ the speed in which the points move along h. Let $w_p > 0$ be the weight (or demand) of a client point p. Given a point u of the plane, let $x(u)$ and $y(u)$ denote the $x-$ and $y-$coordinates of u respectively. The distance or travel time between a point p and the service facility f is given by the function $d_{t,t'}(p,f) := \min \left\{ \|p - f\|_1, \|p - t\|_1 + \frac{\ell}{v} + \|t' - f\|_1, \|p - t'\|_1 + \frac{\ell}{v} + \|t - f\|_1 \right\}$, (see Figure 1).
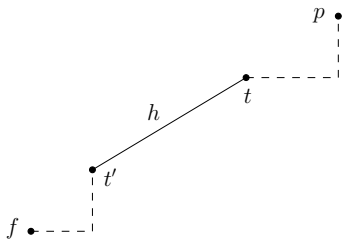

Figure 1: The distance between a point p and the facility f using the highway.

Then the problem can be formulated as follows:

> **The 1-Median and 1-Highway problem (1M1H-problem)**: Given a set S of n points, a weight $w_p > 0$ associated with each point p of S, a fixed highway length $\ell > 0$, and a fixed speed $v \geq 1$, locate a point (facility) f and a line segment (highway) h of length ℓ with endpoints t and t' such that the function $\sum_{p \in S} w_p \cdot d_{t,t'}(p,f)$ is minimized.

Observe that if $\ell = 0$ the 1M1H-problem corresponds to the classical (weighted) rectilinear 1-median problem. In Section 2 we first provide a proper characterization of the solutions. Our proof uses

geometric observations and is simpler than the proof given in [10]. We also give a counterexample to the Espejo and Rodríguez-Chía's characterization. In Section 3 we present an improved algorithm running in $O(n^3)$ time that correctly solves the 1M1H-problem. Finally, in Section 4, we state our conclusions and proposal for further research.

2 Properties of an optimal solution

An easy observation (also stated in [10]) is that the service facility can be located at one of the endpoints of the rapid transit line. From now on, we assume throughout the paper that $f = t'$. This assumption simplifies the distance from a point $p \in S$ to the facility to the following expression:

$$d_t(p, f) = \min\left\{ \|p - f\|_1, \|p - t\|_1 + \frac{\ell}{v} \right\}.$$

Using this observation, the expression of our objective function to minimize is $\Phi(f, t) = \sum_{p \in S} w_p \cdot d_t(p, f)$. We call this value the *total transportation cost* associated with f and t (or simply the *cost* of f and t).

We say that a point p uses the highway if $\|p - t\|_1 + \frac{\ell}{v} < \|p - f\|_1$, and that p does not use it (or goes directly to the facility) otherwise. Given f and t, we call *travel bisector* of f and t (or *bisector*, for short) as the set of points z such that $\|z - f\|_1 = \|z - t\|_1 + \frac{\ell}{v}$, see Figure 2. A geometrical description of such a bisector (boundary of the so-called *captation region*) can be found in [10].
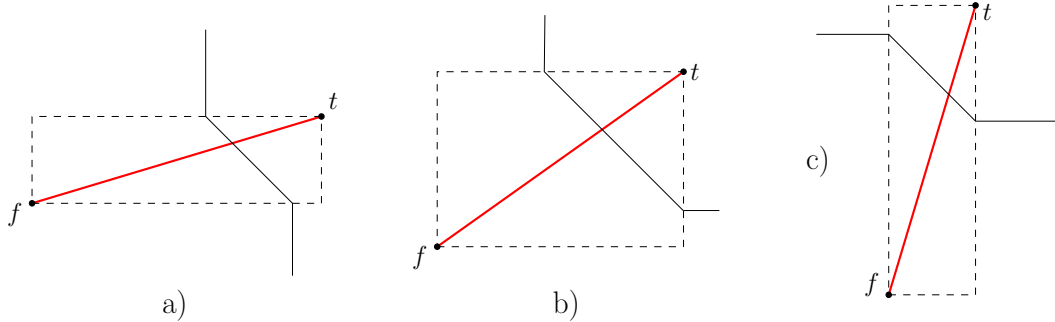

Figure 2: The travel bisector of f and t.

Consider the grid G defined by the set of all axis-parallel lines passing through the elements of S. As mentioned before, Espejo and Rodríguez-Chía [10] claimed that there always exists an optimal solution of the 1M1H-problem for which one of the endpoints of the highway is a vertex of G. Unfortunately, this claim is not true in general. Indeed, Figure 3 shows a counterexample of the claim.

Lemma 2.1 *There exists a problem instance in which no optimal solution to the 1M1H-problem satisfies that one of the endpoints of the highway is a vertex of G, even when all points are assigned unit weight.*

Due to lack of space, proof is omitted. Full details of this claim can be found in [7, 8].

Lemma 2.2 *There exists an optimal solution to the 1M1H-problem satisfying one of the next conditions:*
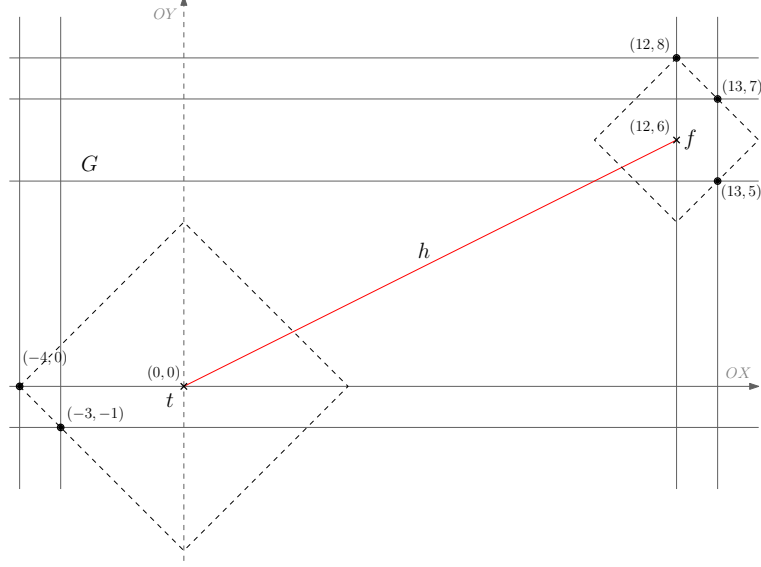
3

Figure 3: Counterexample to Espejo and Rodríguez-Chía [10]'s characterization of the optimal solution. In the problem instance, all points have unit weight.

(a) *One of the endpoints of the highway is a vertex of G.*

(b) *One endpoint of the highway is on a horizontal line of G, and the other endpoint is on a vertical line of G.*

Proof. Let f and t be the endpoints of an optimal highway h and assume that neither of the two conditions is satisfied. Using local perturbation we will transform this solution into one that satisfies one of the two conditions. Assume that neither f nor t is on a vertical line of G. Let $\delta_1 > 0$ (resp. $\delta_2 > 0$) be the smallest value such that if we translate h with vector $(-\delta_1, 0)$ (resp. $(\delta_2, 0)$) then either one endpoint of h touches a vertical line of G or a demand point hits the bisector of f and t. Given $\varepsilon \in [-\delta_1, \delta_2]$, let f_ε, t_ε, and h_ε be f, t, and h translated with vector $(\varepsilon, 0)$, respectively.

While doing the translation, the bisector between t and t' moves. First notice that a point p of S can only change the type of shortest path (from using the highway to walking, or *vice versa*) whenever p crosses the bisector. By the choice of δ_1 and δ_2, this cannot happen during the translation. Moreover, by construction of δ_1 and δ_2, no grid line can pass through a highway endpoint. That is, in the whole translation, the shortest path topology from p to f is also unaffected. Which, by linearity of the L_1 metric, it implies that $|d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)| = \varepsilon$.

Given a real number x, let $\mathrm{sgn}(x)$ denote the sign of x. We partition S into three sets S_1, S_2 and S_3 as follows:

$$
\begin{aligned}
S_1 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = \mathrm{sgn}(\varepsilon), \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\} \\
S_2 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = -\mathrm{sgn}(\varepsilon), \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\} \\
S_3 &= \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = -1, \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\}
\end{aligned}
$$

Geometrically speaking, the elements of S_3 belong to the bisector of f and t, S_1 contains the demand points that travel rightwards to reach f, and S_2 contains the points that travel leftwards. Theoretically, one could consider the case in which a point belongs to set $S_4 = \{p \in S \mid \mathrm{sgn}(d_{t_\varepsilon}(p, f_\varepsilon) - d_t(p, f)) = 1, \quad \forall \varepsilon \in [-\delta_1, \delta_2] \setminus \{0\}\}$. The points of this set are those that, when

4

translating the highway in either direction, the distance between them and the entry point of the highway increases. This situation can only happen when the point is vertically aligned with the entry point: that is, point $p \in S_4$ if and only if either (i) p uses the highway to reach the facility and it is vertically aligned with t, or (ii) p walks to the facility and it is vertically aligned with f. However, by definition of δ_1 and δ_2, no point of S can belong to (or enter) S_4 during the whole translation.

Whenever we translate the highway ε units to the right (for some $0 < \varepsilon \leq \delta_1$), the highway will be ε units closer for points in $S_2 \cup S_3$, but ε units further away for points of S_1. Analogously, the distance to the facility decreases for points in $S_1 \cup S_3$ and increases for points of S_2 when translating h leftwards. Given $X \subseteq S$, let $w(X) = \sum_{p \in X} w_p$ be the sum of weights of the points in set X. Let $N = w(S_1) - w(S_2)$ and $k = w(S_3)$. Thus, the change of the objective function when translating the highway with vector $(\varepsilon, 0)$ (for $\varepsilon \in [-\delta_1, \delta_2]$) is,

$$\sum_{p \in S} w_p \cdot d_{t_\varepsilon}(p, f_\varepsilon) - \sum_{p \in S} w_p \cdot d_t(p, f) = w(S_1)\varepsilon - w(S_2)\varepsilon - w(S_3)|\varepsilon| = N\varepsilon - k|\varepsilon|.$$

We claim that $k = 0$ and $N = 0$. In fact, since h is optimal $N\varepsilon - k|\varepsilon| \geq 0$ for all $\varepsilon \in [-\delta_1, \delta_2]$. If $\varepsilon > 0$ then $N \geq k$. Otherwise, $\varepsilon < 0$ implies $N \leq -k$. Therefore, if $k > 0$ then we have a contradiction, that is, $N \leq -k$ and $N \geq k$. Thus, we have $k = 0$ implying $N = 0$. Therefore we can translate h either rightwards or leftwards in such a way that the objective function keeps unchanged.

More importantly, observe that the value of k must remain 0 on the whole translation. Indeed, if at some point it becomes positive, then we can find a translation from that point that reduces the cost of the objective function. In particular, the set S_3 must remain empty during the whole translation. Any point that changes from set S_1 to S_2 (or *vice versa*) must first enter S_3. Since the latter set remains empty during the whole translation, no point can change between sets $S_1, S_2,$ or S_3 until either f or t is vertically aligned with a point of S.

We repeat this operation until f or t is on a vertical line of G or a point of S reaches the bisector. In the latter case, we will have $k > 0$. In particular, we can find a translation that reduces the total cost of the solution. This contradicts with the assumption that the original highway location was optimal, so it cannot happen. Thus, we can translate h either rightwards or leftwards until one of the two highway endpoints reaches a horizontal line of G. We repeat the same process for the y-coordinates, hence proving the Lemma. $\qquad\square$

When the highway's length is equal to zero, the 1M1H-problem is the weighted 1-median problem in metric L_1 [9], and in this case the item (a) of Lemma 2.2 holds. Lemma 2.2 (a) implies what is stated in [9]: There always exists a weighted 1-median which is a vertex of grid G. It was proved in [10] that the endpoints of any solution of the 1M1H-problem must be in opposite quadrants with respect to some weighted 1-median of S located on a vertex of grid G. Although this property does not help us to reduce the asymptotic time complexity of the algorithm we present in Section 3, it can be used to considerably reduce the solution search space.

In the next section we provide a correct algorithm that solves the problem in $O(n^3)$ time. We assume general position, that is, there are no two points on a same line having slope in the set $\{-1, 0, 1, \infty\}$.

3 The algorithm

Lemma 2.2 can be used to find an optimal solution to the 1M1H-problem. Although the method is quite similar for both cases in Lemma 2.2, we address the two cases independently for the sake of clarity. By Vertex-1M1H-problem we will denote the 1M1H-problem for the cases in which Lemma 2.2 a) holds, and by Edge-1M1H-problem the 1M1H-problem for the cases in which Lemma 2.2 b) holds. In the next subsections we give an $O(n^3)$-time algorithm for each variant of the problem. In both of them we assume w.l.o.g. that the highway length ℓ is equal to one.

In the following θ will denote the positive angle of the highway with respect to the positive direction of the x-axis. For the sake of clarity, we will assume that $\theta \in [0, \frac{\pi}{4}]$. When θ belongs to the interval $[k\frac{\pi}{4}, (k+1)\frac{\pi}{4}]$, $k = 1, \ldots, 7$, both the Vertex- and Edge-1M1H-problem can be solved in a similar way.

Given a point u and an angle θ, let $u(\theta)$ be the point with coordinates $(x(u) + \cos\theta, y(u) + \sin\theta)$. There exists an angle $\phi \in [0, \frac{\pi}{4}]$ such that the bisector of the endpoints f and $t = f(\theta)$ has the shape in Figure 2 a) for all $\theta \in [0, \phi)$, and has the shape in Figure 2 b) for all $\theta \in (\phi, \frac{\pi}{4}]$. Such an angle ϕ verifies $\cos(\phi) - \sin(\phi) = \frac{1}{v}$. Furthermore, $\phi = \frac{1}{2}\arcsin(1 - \frac{1}{v^2})$ and $\phi \neq \frac{\pi}{4}$ unless v is infinite.

Let Π_x, Π_y, and Π_{x+y} denote set S sorted according to the $x-$, $y-$, and $(x+y)-$ order, respectively.

3.1 Solving the Vertex-1M1H-problem

For each vertex u of G we can solve the problem subject to $f = u$ or $t = u$. We show how to obtain a solution if $f = u$. The case where $t = u$ can be solved analogously. Suppose w.l.o.g. that the vertex $f = u$ is the origin of the coordinate system and the highway angle is θ, for $\theta \in [0, \frac{\pi}{4}]$. Thus, the weighted distance between a point $p \in S$ and the facility u has the expression $c_1 + c_2\cos\theta + c_3\sin\theta$, where $c_1 > 0$ and either $c_2, c_3 = \pm w_p$ (p uses the highway) or $c_2 = c_3 = 0$ (p does not use the highway). When θ goes from 0 to $\frac{\pi}{4}$ this expression changes at the values of θ such that:

- The point p switches from using the highway to going directly to the facility (or vice versa). We call these changes *bisector events*. A bisector event occurs when the bisector between the highway's endpoints u and $u(\theta)$, contains p. At most two bisector events are obtained for each point p.

- The highway endpoint $u(\theta)$ crosses the vertical or horizontal line passing through p. We call this event *grid event*. Again, each point of S generates at most two grid events.

- $\theta = \phi$. This event is called ϕ-*event*.

Lemma 3.1 *After an $O(n\log n)$-time preprocessing, the angular order of all the events associated with a given vertex of G can be obtained in linear time.*

Proof. The preprocessing consists in computing Π_x, Π_y, and Π_{x+y}, which can be done in $O(n\log n)$ time. Now, let u be a vertex of G. It is straightforward to see that there are $O(n)$ grid events and that we can obtain their angular order in linear time by using both Π_x and Π_y. Let us show how to obtain the bisector events in $O(n)$ time.

The bisector of u and $u(\theta)$ consists of two axis-aligned half-lines and a line segment with slope -1 connecting their endpoints (see Figure 2 and [10] for further details). Given a point p, when θ goes from 0 to $\pi/4$ the bisector between u and $u(\theta)$ passes through p at most twice, that is, when

p belongs to one of the half-lines of the bisector and when p belongs to the line segment. If p belongs to the line segment of the bisector then the event is denoted by α_p (see Figure 4 b)). If p belongs to the leftmost half-line of the bisector, which is always vertical, we denote that event by β_p (see Figure 4 a)). Otherwise, if p belongs to the rightmost half-line which can be either vertical or horizontal we denote that event by γ_p (see Figure 4 c) and d)). Observe that if the rightmost half-line is vertical then $\gamma_p < \phi$, otherwise $\gamma_p > \phi$.
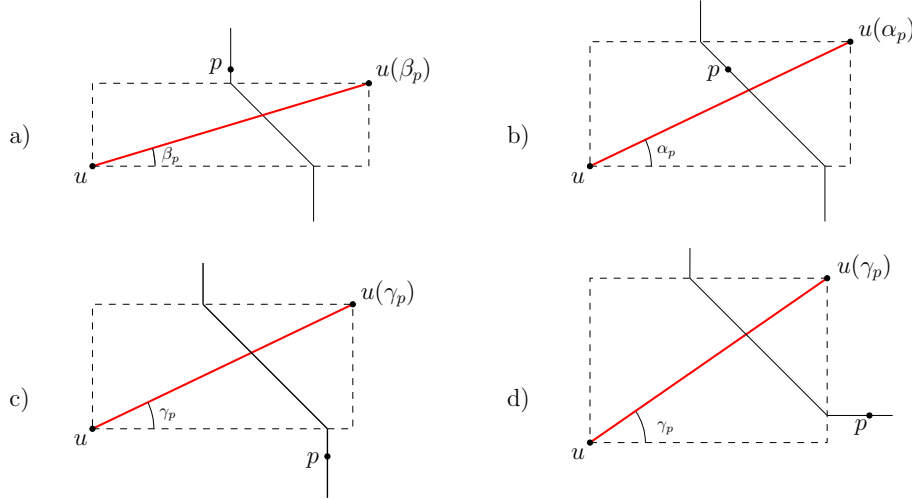

Figure 4: The bisector events of p when $\theta \in [0, \frac{\pi}{4}]$. a) p belongs to the leftmost half-line of the bisector of u and $u(\theta)$. b) p belongs to the segment. c–d) p belongs to the rightmost half-line of the bisector.

Let Π_1 be the subsequence of Π_{x+y} containing all elements p such that $\alpha_p \in [0, \frac{\pi}{4}]$, Π_2 be the subsequence of Π_x containing all elements p such that $\beta_p \in [0, \frac{\pi}{4}]$, and Π_3 be the subsequence of Π_x that contains all elements p such that $y(p) < y(u)$ and $\gamma_p \in [0, \frac{\pi}{4}]$, concatenated with the subsequence of Π_y that contains all elements p such that $x(p) > x(u)$ and $\gamma_p \in [0, \frac{\pi}{4}]$. Given a point $p \in S$, the corresponding events of p in $[0, \frac{\pi}{4}]$ can be found in constant time, thus Π_1, Π_2, and Π_3 can be built in linear time. The following statements are true for any point $p \in S$:

(a) $x(p) + y(p) = \frac{1}{2}(\cos\alpha_p + \sin\alpha_p + \frac{1}{v})$ for all points p in Π_1.

(b) $x(p) = \frac{1}{2}(\cos\beta_p - \sin\beta_p + \frac{1}{v})$ for all points p in Π_2.

(c) $x(p) = \frac{1}{2}(\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p < \phi$.

(d) $y(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p > \phi$.

Let Γ_1 (resp. Γ_2, Γ_3) be the sequence obtained by replacing each element p in Π_1 (resp. Π_2, Π_3) by α_p (resp. β_p, γ_p). Therefore, from statements $(a) - (d)$ and the monotonicity of the functions $\cos\theta + \sin\theta$, $\cos\theta - \sin\theta$, and $-\cos\theta + \sin\theta$ in the interval $[0, \frac{\pi}{4}]$, we obtain that Γ_1, Γ_2, and Γ_3 are sorted sequences. Using a standard method for merging sorted lists, we can merge in linear time Γ_1, Γ_2, Γ_3, the grid events, and the ϕ-event. Therefore, the angular order of all events associated with a vertex u can be obtained in $O(n)$ time and the result follows. $\qquad\square$

Theorem 3.2 *The Vertex-1M1H-problem can be solved in $O(n^3)$ time.*

Proof. Let u be a vertex of G. Using Lemma 3.1, we obtain in linear time the angular order of the $O(n)$ events associated with u. The events induce a partition of $[0, \frac{\pi}{4}]$ into maximal intervals.

For each of those intervals, the objective function takes the form $g(\theta) := \sum_{p \in S} w_p \cdot d_t(p, u) = b_1 + b_2 \cos\theta + b_3 \sin\theta$. This problem is of constant size in each subinterval and the minimum of $g(\theta)$ can be found in $O(1)$ time. Furthermore, the expression of $g(\theta)$ can be updated in constant time when θ crosses an event point distinct of ϕ when going from 0 to $\frac{\pi}{4}$. In the case where θ crosses ϕ, $g(\theta)$ can be updated in at most $O(n)$ time. Then the problem subject to $f = u$ can be solved in linear time. The case in which $t = u$ can be addressed in a similar way. It gives an overall $O(n^3)$ time complexity because G has $O(n^2)$ vertices. $\qquad \square$

3.2 Solving the Edge-1M1H-problem

We now consider the case in which the optimal solution satisfies condition b) of Lemma 2.2. Namely, we consider a horizontal line e_h of G and each vertical line e_v of G. For every pair of such lines, we consider eight different sub-cases, depending on whether h is located above/below e_h, rightwards/leftwards of e_v, and $f \in e_h$ and $t \in e_v$ (or *vice versa*). For a fixed sub-case, we parameterize the location of the highway by the angle θ that the highway forms with e_h. As in the Vertex-1M1H case, we assume that $f \in e_h$, $t \in e_v$, and $\theta \in [0, \frac{\pi}{4}]$.

We implicitly redefine the coordinate system so that e_h and e_v intersect at the origin o. Let $\theta \in [0, \frac{\pi}{4}]$ be the positive angle of the highway with respect to the positive direction of the x-axis and $f = x_\theta$, $t = y_\theta$ be the highway endpoints.

First notice that, since we are again doing a continuous translation of h, the events that affect the value of the objective function are exactly the same as those that happen in the Vertex-1M1H-problem: bisector-, grid- and ϕ- events. We start by showing that the equivalent of Lemma 3.1 also holds:

Lemma 3.3 *After an $O(n \log n)$-time preprocessing, the angular order of all the events associated with a pair of perpendicular lines of G can be obtained in linear time.*

Proof. We can follow the arguments of Lemma 3.1. Firstly, we note that there are $O(n)$ grid events and their angular order can be obtained in linear time by using both Π_x and Π_y.

Given a point $p \in S$, let the events α_p, β_p, and γ_p be defined as in the Vertex-1M1H case. Refer to Figure 4. Let Π_1 be the subsequence of Π_{x+y} containing all elements p such that $\alpha_p \in [0, \frac{\pi}{4}]$, Π_2 be the subsequence of Π_x containing all elements p such that $\beta_p \in [0, \frac{\pi}{4}]$, and Π_3 be the subsequence of Π_x that contains all elements p such that $y(p) < y(o)$ and $\gamma_p \in [0, \phi]$, concatenated with the subsequence of Π_y that contains all elements p such that $x(p) > x(o)$ and $\gamma_p \in [\phi, \frac{\pi}{4}]$. Note that Π_1, Π_2, and Π_3 can be built in linear time.

Given a point $p \in S$, the following statements are true:

(a) $x(p) + y(p) = \frac{1}{2}(-\cos\alpha_p + \sin\alpha_p + \frac{1}{v})$ for all points p in Π_1.

(b) $x(p) = \frac{1}{2}(-\cos\beta_p - \sin\beta_p + \frac{1}{v})$ for all points p in Π_2.

(c) $x(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p < \phi$.

(d) $y(p) = \frac{1}{2}(-\cos\gamma_p + \sin\gamma_p + \frac{1}{v})$ for all points p in Π_3 such that $\gamma_p > \phi$.

Let Γ_1 (resp. Γ_2, Γ_3) be the sequence obtained by replacing each element p in Π_1 (resp. Π_2, Π_3) by α_p (resp. β_p, γ_b). Therefore, by using similar arguments to those used in Lemma 3.1 the angular order of all events can be obtained in $O(n)$ time, once the lists Π_x, Π_y and Π_{x+y} have been precomputed. $\qquad \square$

Consider now a small interval $[\theta_1, \theta_2]$ in which no event occurs. After the coordinate system re-definition, we have $f = x_\theta = (-\cos\theta, 0)$, and $t = y_\theta = (0, \sin\theta)$. Let $p \in S$ be a point that uses the highway to reach the facility; since only the y-coordinate of t changes, its distance to f can be expressed as $c_1 \pm \sin\theta$ for some $c_1 > 0$. Analogously, if p walks to f, its distance is of the form $c_1 \pm \cos\theta$ for some $c_1 > 0$. That is, the distance between a point of S and f in any interval is of the form $c_1 + c_2 \sin\theta + c_3 \cos\theta$ for some constants $c_1 > 0$ and $c_2, c_3 \in \{-1, 0, 1\}$.

Theorem 3.4 *The Edge-1M1H-problem can be solved in $O(n^3)$ time.*

Proof. We can use a method similar to the one used in the Vertex-1M1H-problem. Let e_h be a horizontal line of G and e_v be a vertical line of G.

Using Lemma 3.3, we obtain in linear time the angular order of the $O(n)$ events associated with e_h and e_v. The events induce a partition of $[0, \frac{\pi}{4}]$ into maximal intervals. For each of those intervals the objective function has the form $g(\theta) := \Phi(f, t) = \Phi(x_\theta, y_\theta) = b_1 + b_2 \cos\theta + b_3 \sin\theta$, where $b_1 > 0$, and $b_2, b_3 \in \mathbb{Z}$ are constants. This problem has constant size, hence the minimum of $g(\theta)$ can be found in $O(1)$ time. Furthermore, the expression of $g(\theta)$ can be updated in constant time when θ crosses an event point distinct of ϕ when it goes from 0 to $\frac{\pi}{4}$. In the case where θ crosses ϕ, $g(\theta)$ can be updated in at most $O(n)$ time. Then the problem subject to $f \in e_h$ and $t \in e_v$ can be solved in linear time. It gives an overall $O(n^3)$ time complexity because G has $O(n^2)$ pairs consisting of a horizontal and a vertical line. $\square$

4 Conclusions and Further Research

In this paper we have addressed a facility location problem in the plane in which a single service facility and a rapid transit line are simultaneously located in order to minimize the total travel time of the clients to the facility, under the L_1 metric. In addition to correcting the error in characterization of [10], we provide a faster method for computing the solution to the the 1-Median and 1-Highway problem that also considers the disregarded case of Espejo and Rodríguez-Chía.

Although, extensive work has been done in highway location problems (see a small summary in Section 1), this is the first algorithm in which a highway of arbitrary orientation can be located (other than the rotating calipers approach of [2] and the $1 + \epsilon$-approximation of [5]). A natural question if this approach can be extended to similar problems.

More interestingly, the 1M1H-problem suggests a new collection of open problems in the facility location area. The first question is about the general pMkH-problem, where p facilities and k highways have to be located in order to minimize the total travel cost. It is easy to see that the problem is NP-complete when either p or k is part of the input [12, 13], even when the other one is set to zero. However, it would be interesting to find if some FPT or approximation algorithm can be designed.

The 1M1H-problem can also be generalized in other directions. For example, we could consider a similar distance model in which the clients can enter and exit the highway at any point (called *freeway* in [4]). Another variation could be studying the same problem in other metrics or using different optimization criteria. By using the minimax criterion, the unweighted case has been solved in $O(n^2)$ time in [6]. However, no results are known for the general problem.

From the experimental results of [10, 8] we can deduce that the highway's length has a strong impact on the optimal solution. Hence, considering a variation of the problem in which the length of the

highway is not given in advance, and should be somehow decided by the algorithm is another possible direction. Specially, one would like to somehow find a balance between the cost of constructing a longer highway and the improvement in the total transportation cost.

5   Acknowledgements

References

[1] M. Abellanas, F. Hurtado, C. Icking, R. Klein, E. Langetepe, L. Ma, B. Palop, and V. Sacristán. Voronoi diagram for services neighboring a highway. *Information Processing Letters*, 86:283–288, 2003.

[2] H.-K. Ahn, H. Alt, T. Asano, S. W. Bae, P. Brass, O. Cheong, C. Knauer, H.-S. Na, C.-S. Shin, and A. Wolff. Constructing optimal highways. In *Proceedings of the 13th Computing: The Australasian Theory Symposium (CATS'07)*, pages 7–14, 2007.

[3] O. Aichholzer, F. Aurenhammer, and B. Palop. Quickest paths, straight skeletons, and the city Voronoi diagram. *Discrete & Computational Geometry*, 31:17–35, 2004.

[4] S. W. Bae, M. Korman, and T. Tokuyama. All farthest neighbors in the presence of highways and obstacles. In *Proceedings of the 3rd International Workshop on Algorithms and Computation (WALCOM'09)*, pages 71–82, 2009.

[5] J. Cardinal, S. Collette, F. Hurtado, S. Langerman, and B. Palop. Optimal location of transportation devices. *Computational Geometry: Theory and Applications*, 41:219–229, 2008.

[6] J.-M. Díaz-Báñez, M. Korman, P. Pérez-Lantero, and I. Ventura. The 1-center and 1-highway problem. In *Proc. of the 14th Spanish Meeting on Computational Geometry*, pages 193–196, 2011.

[7] J.-M. Díaz-Báñez, M. Korman, P. Pérez-Lantero, and I. Ventura. Locating a service facility and a rapid transit line. In *Proc. of the 14th Spanish Meeting on Computational Geometry*, pages 189–192, 2011.

[8] J.-M. Díaz-Báñez, M. Korman, P. Pérez-Lantero, and I. Ventura. Locating a service facility and a rapid transit line. *CoRR*, abs/1104.0753, 2011.

[9] R. Durier and C. Michelot. Geometrical properties of the fermat-weber problem. *European Journal of Operational Research*, 20(3):332–343, 1985.

[10] I. Espejo and A. M. Rodríguez-Chía. Simultaneous location of a service facility and a rapid transit line. *Computers and Operations Research*, 38:525–538, February 2011.

[11] I. Espejo and A. M. Rodríguez-Chía. Corrigendum to "simultaneous location of a service facility and a rapid transit line" [computers & operations research 38(2) (2011) 525-538]. *Computers & OR*, 39(11):2899–2903, 2012.

[12] T. Feder and D. Greene. Optimal algorithms for approximate clustering. In *Proceedings of the twentieth annual ACM symposium on Theory of computing*, STOC '88, pages 434–444, New York, NY, USA, 1988. ACM.

[13] M. Korman. *Theory and Applications of Geometric Optimization Problems in Rectilinear Metric Spaces*. PhD thesis, Tohoku University, 2009.

[14] M. Korman and T. Tokuyama. Optimal insertion of a segment highway in a city metric. In *Proceedings of the 14th international conference on Computing and Combinatorics (COCOON'08)*, pages 611–620, 2008.