Finding all pure strategy Nash equilibria in a planar location game

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Abstract

In this paper, we deal with a planar location-price game where firms first select their locations and then set delivered prices in order to maximize their profits. If firms set the equilibrium prices in the second stage, the game is reduced to a location game for which pure strategy Nash equilibria are studied assuming that the marginal delivered cost is proportional to the distance between the customer and the facility from which it is served. We present characterizations of local and global Nash equilibria. Then an algorithm is shown in order to find all possible Nash equilibrium pairs of locations. The minimization of the social cost leads to a Nash equilibrium. An example shows that there may exist multiple Nash equilibria which are not minimizers of the social cost.

Keywords: Location, Game theory, Nash equilibrium.

1 Introduction

Major decisions for firms selling the same type of product and competing for customers are where to locate their facilities and what price to set. When the firms use delivered pricing, profit is strongly affected by both the location of their facilities and the price they set in each demand point. The maximization of profit for each competing firm can be seen as a location-price game, which has been studied since the work by Hotelling [19]. Many existing literature deals with this type of problem on a linear location space (see [4], [14], [22]), which is in part due to the complexity of solving the associated location problems in other location spaces as the plane or a network (see the survey papers [12], [13], [25], [26]).

The existence of a price equilibrium was shown for the first time by Hoover [18], who analyzed spatial discriminatory pricing for firms with fixed locations and concluded that a firm serving a particular market would be constrained in its local price by the delivery cost of the other firms serving that market. In situations where demand elasticity is 'not too high', the equilibrium price at a given market is equal to the delivery cost of the firm with the second lowest delivery cost. This result was extended later to spatial duopoly (see [20], [21]) and to spatial oligopoly for different types of location spaces (see [8], [15]).

If firms set the equilibrium prices, which are determined by their facility locations, the location-price game reduces to a location game. This location game has been scarcely studied in the location

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literature. In a duopoly with completely inelastic demand and constant marginal production costs, Lederer and Thisse [21] showed that a location equilibrium exists which is a global minimizer of the *social cost*. The social cost is defined as the total delivered cost if each customer were served with the lowest marginal delivered cost. In oligopoly, the same result is obtained by Dorta et al. [8], who present a model where firms make location and delivery price decisions along a network of connected but spatially separated markets.

The problem of finding Nash equilibria to the location game has been solved by *social cost minimization*. This optimization problem is equivalent to the *p-median problem* when the marginal delivered cost from each site location to each demand point is the same for all competing firms. There is an extensive literature on algorithms to solve the p-median problem for both network and planar location spaces (see for instance [1, 9, 27]). If marginal delivered cost from each site location to each demand point is different for each competitor the social cost minimization problem has been solved on a network location space by using a Mixed Integer Linear Programming (*MILP*) formulation (see [23]). If demand is price sensitive or marginal production costs are not constant, the socially optimal locations may not be an equilibrium of the location game which has been shown in [16] and [17]. Other similar location games where the payoffs are given by market share have been recently studied in [2, 3] and [10].

In this paper, we study the location game for two firms in a two-dimensional location space assuming that the marginal delivered cost is proportional to the distance between customer and facility from which it is served. In Section 2 we state the problem. In Section 3 we prove the existence of at least a Nash equilibrium. In Section 4 some useful properties and characterizations of local and global Nash equilibria are presented. In Section 5, we propose an algorithm to generate all Nash equilibria to the location game, and show some illustrative examples. It is also shown the existence of Nash equilibria which are not minimizers of social cost. Finally, some conclusions are presented in Section 6.

2 The problem

We consider the location-price game in a planar space for two competing firms, Firm 1 and Firm 2. For simplicity we assume that the marginal delivered cost is given by the distance between the customer and the facility from which it is served. The Euclidean metric is taken to measure the distance between any pair of points. Thus $d(x, y)$ denotes the Euclidean distance between the points x and y. Customers are located in different points in the plane, we denote by $P = \{p_1, p_2, \ldots, p_n\}$ the set of demand points representing the customers. They have fixed demands for an homogeneous product. Let $q_i > 0$ be the demand at the point p_i, $1 \leq i \leq n$. Each customer buys from the facility offering the lowest price.

It is assumed that demand is completely inelastic at each node. Each firm locates a single facility and set a delivered price at each demand point which is the equilibrium price resulting from the price competition process. Let x and y be any fixed pair of locations for the competing firms. The equilibrium price at each demand point p_i for Firm 1 and Firm 2 are respectively:

$$P_i^1(x,y) = \begin{cases} d(p_i, y) & \text{if} \quad d(p_i, x) < d(p_i, y) \\ d(p_i, x) & \text{otherwise} \end{cases}$$

$$P_i^2(x,y) = \begin{cases} d(p_i, x) & \text{if} \quad d(p_i, y) < d(p_i, x) \\ d(p_i, y) & \text{otherwise} \end{cases}$$

Thus, the demand at each point is captured by its closest facility. Then, for each pair of facility location points x and y, which correspond to the locations of the two competing facilities, the partition induced by the perpendicular bisector of the segment connecting both points determines the customers captured by each firm. We denote as $[x, y]$ the line segment connecting x and y.

Throughout this paper, we will use the following notation. Let $I = \{1, 2, 3, ..., n\}$ be the set of indexes of the points in P. Given a pair of points x, y in $\mathbb{R}^2$, we will denote by $I_{x|y}$, $I_{y|x}$, and I_{xy}, the subsets of indexes $I_{x|y} = \{i \in I : d(p_i, x) < d(p_i, y)\}$, $I_{y|x} = \{i \in I : d(p_i, x) > d(p_i, y)\}$, and $I_{xy} = \{i \in I : d(p_i, x) = d(p_i, y)\}$, respectively. For an $\varepsilon > 0$, $\mathbb{B}(x, \varepsilon)$ will denote the set $\{x' \in \mathbb{R}^2 : d(x, x') < \varepsilon\}$. Given a subset $J \subseteq I$, the point set $\{p_i : i \in J\}$ is sometimes identified with the set of indexes J.

The profit obtained by the facility x corresponding to Firm 1 is given by the function

$$\pi_1(x, y) = \sum_{i \in I_{x|y}} (d(p_i, y) - d(p_i, x)) q_i$$

and the profit gained by the facility y corresponding to Firm 2 is given by

$$\pi_2(x, y) = \sum_{j \in I_{y|x}} (d(p_j, x) - d(p_j, y)) q_j.$$

Note that the customers located on the bisector of the segment connecting the points x and y do not provide profit to any facility. The location problem is stated as a non-cooperative game with simultaneous location decisions and payoff given by the profit obtained by each player.

Proposition 1 *The profit functions π_1 and π_2 have the following properties:*

(a) π_1 and π_2 are continuous functions in $\mathbb{R}^2 \times \mathbb{R}^2$.

(b) $\pi_1(y, x) = \pi_2(x, y)$ and $\pi_2(y, x) = \pi_1(x, y)$, for any $x, y \in \mathbb{R}^2$.

(c) Given $a, b \in \mathbb{R}^2$ there is $M > 0$ such that if $\|x\| > M$ then $\pi_2(a, x) = 0$ and $\pi_1(x, b) = 0$.

(d) $\pi_1(x, y) - \pi_2(x, y) = \sum_{i \in I} d(p_i, y) q_i - \sum_{i \in I} d(p_i, x) q_i$.

Proof. Properties (a), (b) and (c) are straightforward. We prove only property (d).

$$\begin{aligned}
\pi_1(x, y) - \pi_2(x, y) &= \sum_{i \in I_{x|y}} (d(p_i, y) - d(p_i, x)) q_i - \sum_{j \in I_{y|x} \cup I_{xy}} (d(p_j, x) - d(p_j, y)) q_j \\
&= \sum_{i \in I_{x|y}} (d(p_i, y) - d(p_i, x)) q_i + \sum_{j \in I_{y|x} \cup I_{xy}} (d(p_j, y) - d(p_j, x)) q_j \\
&= \sum_{i \in I} d(p_i, y) q_i - \sum_{i \in I} d(p_i, x) q_i.
\end{aligned}$$

□

Local and global equilibria for the set of demand points P are defined as follows:

Definition 1 *The pair of points $(\bar{x}, \bar{y})$ is a local equilibrium for the set P if there exists $\varepsilon > 0$ such that:*

$$\begin{aligned}
\pi_1(\bar{x}, \bar{y}) &\geq \pi_1(x, \bar{y}), \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon) \\
\pi_2(\bar{x}, \bar{y}) &\geq \pi_2(\bar{x}, y), \quad \forall y \in \mathbb{B}(\bar{y}, \varepsilon).
\end{aligned}$$

Definition 2 *The pair of points* $(\bar{x}, \bar{y})$ *is a global (or Nash) equilibrium for the set* P *if:*

$$\begin{aligned} \pi_1(\bar{x}, \bar{y}) &\geq \pi_1(x, \bar{y}), \quad \forall x \in \mathbb{R}^2 \\ \pi_2(\bar{x}, \bar{y}) &\geq \pi_2(\bar{x}, y), \quad \forall y \in \mathbb{R}^2. \end{aligned}$$

Note that every global equilibrium is also a local one. In the following, we will deal with the problem of finding all Nash equilibrium pairs of points for the location game considering the previous profit functions.

3 Existence of Nash equilibria

In this section, we introduce the *social cost* function in order to prove that there exists at least one global equilibrium for the proposed location game.

Definition 3 *Given a pair of points* x, y *in* $\mathbb{R}^2$, *its social cost function is defined by*

$$S(x, y) = \sum_{i \in I} \min\{d(p_i, x), d(p_i, y)\} q_i.$$

The social cost function has the following properties:

Proposition 2 $S(x, y)$ *is continuous in* $\mathbb{R}^2 \times \mathbb{R}^2$, *and there is a pair of points* $\bar{x}$, $\bar{y}$ *such that* $S(\bar{x}, \bar{y}) \leq S(x, y)$ *for all* x, y *in* $\mathbb{R}^2$.

Proof. Let $CH(P)$ be the convex hull of P. Given x, y in $\mathbb{R}^2$ there are two points x', y' in $CH(P)$ such that

$$d(p_i, x') \leq d(p_i, x) \text{ and } d(p_i, y') \leq d(p_i, y), \text{ for } i = 1, 2, \ldots, n.$$

Hence $S(x', y') \leq S(x, y)$. The function $S(x, y)$ is continuous in the compact set $CH(P) \times CH(P)$, therefore there is a pair $(\bar{x}, \bar{y})$ in $CH(P) \times CH(P)$ that minimizes the function $S(x, y)$. □

Proposition 3 *The profit and social cost functions satisfy:*

$$\pi_1(x, y) = \sum_{i \in I} d(p_i, y) q_i - S(x, y), \qquad \pi_2(x, y) = \sum_{i \in I} d(p_i, x) q_i - S(x, y).$$

Proof. The social cost function can be expressed as follows:

$$S(x, y) = \sum_{i \in I_{x|y}} d(p_i, x) q_i + \sum_{i \in I_{y|x} \cup I_{xy}} d(p_i, y) q_i$$

Then we have:

$$\begin{aligned} \pi_1(x, y) &= \sum_{i \in I_{x|y}} (d(p_i, y) - d(p_i, x)) q_i \\ &= \sum_{i \in I_{x|y}} d(p_i, y) q_i - \sum_{i \in I_{x|y}} d(p_i, x) q_i \\ &= \sum_{i \in I_{x|y}} d(p_i, y) q_i + \sum_{i \in I_{y|x} \cup I_{xy}} d(p_i, y) q_i - \sum_{i \in I_{y|x} \cup I_{xy}} d(p_i, y) q_i - \sum_{i \in I_{x|y}} d(p_i, x) q_i \\ &= \sum_{i \in I} d(p_i, y) q_i - S(x, y). \end{aligned}$$

A similar expression can be obtained for the profit function π_2. □

Proposition 4 *Any global minimizer of $S(x,y)$ is a global equilibrium for P.*

Proof. Let $(\bar{x},\bar{y})$ be a global minimizer of $S(x,y)$. Then, for all x in $\mathbb{R}^2$ we have:

$$\pi_1(\bar{x},\bar{y}) = \sum_{i\in I} d(p_i,\bar{y})q_i - S(\bar{x},\bar{y}) \geq \sum_{i\in I} d(p_i,\bar{y})q_i - S(x,\bar{y}) = \pi_1(x,\bar{y}).$$

Similarly, we obtain that $\pi_2(\bar{x},\bar{y}) \geq \pi_2(\bar{x},y)$ for all y in $\mathbb{R}^2$. Therefore, $(\bar{x},\bar{y})$ is a global equilibrium for P. □

We notice that the claim of Proposition 4 is also valid for any metric space. The converse of Proposition 4 is not true in general. In Subsection 5.1 an example is shown for which there exist global equilibria which are not social cost minimizers.

From Propositions 2 and 4 we obtain the following result:

Proposition 5 *There always exists a Nash equilibrium for P.*

4 Properties of local and global equilibria

Before describing the procedure to find all Nash equilibria, in this section we give some properties to discretize the search space for equilibrium positions.

Definition 4 *A point w in $\mathbb{R}^2$ is a Weber point of a given subset $J \subseteq I$ if and only if*

$$\sum_{j\in J} d\left(p_j,w\right)q_j \leq \sum_{j\in J} d\left(p_j,x\right)q_j, \quad \forall x\in\mathbb{R}^2$$

that is, w is a 1-median point of the set $\{p_j : j\in J\}$.

Definition 5 *A subset $J\subseteq I$ is a candidate subset of P if the sets $\{p_i\in P : i\in J\}$ and $\{p_i \in P : i\in I\backslash J\}$ are separable by a straight line.*

Given two points x and y, let suppose that I_{xy} is empty. Then, both subsets $I_{x|y}$ and $I_{y|x}$ are candidates since they are separable by the bisector of the segment joining x and y, $[x,y]$. Otherwise, if I_{xy} is not empty, then the subsets $I_{x|y}\cup I_{xy}$ and $I_{y|x}$ are also candidates by considering a parallel straight line close enough to the bisector of $[x,y]$. Analogously for the sets $I_{x|y}$ and $I_{y|x}\cup I_{xy}$.

Definition 6 *The Weber set of P, denoted by W, is the set of Weber points corresponding to the candidate subsets of P.*

In the following we will show that the search for equilibrium pairs can be reduced to the set $W\times W$. Before that, we give a technical lemma on candidate sets that will be useful later.

Lemma 6 *Given a demand point set P and two points $\bar{x},\bar{y}$ in $\mathbb{R}^2$, we have:*

(1) There exists $\varepsilon > 0$ such that $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$ for all $x\in\mathbb{B}(\bar{x},\varepsilon)$.

(2) There exists $\varepsilon' > 0$ such that $I_{\bar{y}|\bar{x}} \subseteq I_{y|\bar{x}} \subseteq I_{\bar{y}|\bar{x}} \cup I_{\bar{x}\bar{y}}$ for all $y\in\mathbb{B}(\bar{y},\varepsilon')$.

Proof. We prove only statement (1). The proof of statement (2) is similar. Let $\bar{y}$ be fixed and for each point $p_i \in P$ define ε_i as follows:

$$\varepsilon_i = \begin{cases} d(p_i,\bar{y}) - d(p_i,\bar{x}) & \text{if } \ p_i \in I_{\bar{x}|\bar{y}} \\ d(p_i,\bar{x}) - d(p_i,\bar{y}) & \text{if } \ p_i \in I_{\bar{y}|\bar{x}} \\ +\infty & \text{otherwise} \end{cases}$$

Let ε be a positive value less than $\min\{\varepsilon_i\}$, $x \in \mathbb{B}(\bar{x},\varepsilon)$, and $p_i \in P$. On one hand, if $p_i \in I_{\bar{x}|\bar{y}}$ then:

$$\begin{aligned} d(p_i,x) &\leq d(x,\bar{x}) + d(p_i,\bar{x}) \\ &< \varepsilon + d(p_i,\bar{x}) \\ &< \varepsilon_i + d(p_i,\bar{x}) \\ &= d(p_i,\bar{y}) - d(p_i,\bar{x}) + d(p_i,\bar{x}) \\ &= d(p_i,\bar{y}) \end{aligned}$$

Thus $p_i \in I_{x|\bar{y}}$ and therefore $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}}$. On the other hand, if $p_i \in I_{\bar{y}|\bar{x}}$ then:

$$\begin{aligned} d(p_i,x) &\geq d(p_i,\bar{x}) - d(x,\bar{x}) \\ &> d(p_i,\bar{x}) - \varepsilon \\ &> d(p_i,\bar{x}) - \varepsilon_i \\ &= d(p_i,\bar{x}) - (d(p_i,\bar{x}) - d(p_i,\bar{y})) \\ &= d(p_i,\bar{y}) \end{aligned}$$

It means that $I_{\bar{y}|\bar{x}} \subseteq I_{\bar{y}|x}$ which is equivalent to $I_{x|\bar{y}} \cup I_{x\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$, implying that $I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$. Hence, the result follows. □

Lemma 7 *Let $(\bar{x},\bar{y})$ be a pair of points and $\{J_1, J_2\}$ be any partition of $I_{\bar{x}\bar{y}}$ induced by a straight line. The following statements hold:*

(1) If $\varepsilon > 0$ satisfies condition (1) of Lemma 6, then there exist $x, x' \in \mathbb{B}(\bar{x},\varepsilon)$ such that $I_{x|\bar{y}} = I_{\bar{x}|\bar{y}} \cup J_1$ and $I_{x'|\bar{y}} = I_{\bar{x}|\bar{y}} \cup J_2$.

(2) If $\varepsilon' > 0$ satisfies condition (2) of Lemma 6, then there exist $y, y' \in \mathbb{B}(\bar{y},\varepsilon')$ such that $I_{y|\bar{x}} = I_{\bar{y}|\bar{x}} \cup J_1$ and $I_{y'|\bar{x}} = I_{\bar{y}|\bar{x}} \cup J_2$.

Proof. We prove the existence of x in statement (1). The rest of the proof is similar. Let $\{J_1, J_2\}$ be a partition of $I_{\bar{x}\bar{y}}$ induced by the straight line ℓ, q be the intersection point of ℓ and the bisector $\ell_{\bar{x}\bar{y}}$ of $[\bar{x},\bar{y}]$, and $\varepsilon > 0$ satisfy condition (1) of Lemma 6. It suffices to prove that there exists $x \in \mathbb{B}(\bar{x},\varepsilon) \setminus \{\bar{x}\}$ such that the bisector of $[x,\bar{y}]$ passes through q. It is as follows. Denote by $x_\ell \in \mathbb{R}^2$ the point such that ℓ is the bisector of $[x_\ell,\bar{y}]$. If we rotate ℓ with center at q in order to reach $\ell_{\bar{x}\bar{y}}$, then x_ℓ tends to $\bar{x}$. Then there is an instant in which $x_\ell \in \mathbb{B}(\bar{x},\varepsilon) \setminus \{\bar{x}\}$. By taking $x = x_\ell$ at this instant the result follows. □

Proposition 8 *Let $\bar{x}$ and $\bar{y}$ be two points in $\mathbb{R}^2$. The next statements hold:*

(a) If there exists $\varepsilon > 0$ such that $\pi_1(\bar{x},\bar{y}) \geq \pi_1(x,\bar{y})$ for all $x \in \mathbb{B}(\bar{x},\varepsilon)$, then $\bar{x}$ is a Weber point of $I_{x|\bar{y}}$ for all x such that $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$.

(b) If there exists $\varepsilon' > 0$ such that every $y \in \mathbb{B}(\bar{y},\varepsilon')$ satisfies $\pi_2(\bar{x},\bar{y}) \geq \pi_2(\bar{x},y)$, then $\bar{y}$ is a Weber point of $I_{y|\bar{x}}$ for all y such that $I_{\bar{y}|\bar{x}} \subseteq I_{y|\bar{x}} \subseteq I_{\bar{y}|\bar{x}} \cup I_{\bar{x}\bar{y}}$.

Proof. We prove only the first part of the proposition. The second one is analogous. Let $\varepsilon_1 > 0$ such that $\pi_1(\bar{x}, \bar{y}) \geq \pi_1(x, \bar{y})$ for all $x \in \mathbb{B}(\bar{x}, \varepsilon_1)$, $\varepsilon_2 > 0$ satisfy condition (1) of Lemma 6, and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. On one hand, by Lemma 7, every set $I_{x|\bar{y}}$ such that x satisfies $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$ can be obtained by taking some point $x \in \mathbb{B}(\bar{x}, \varepsilon)$. On the other hand, we have the following for all $x \in \mathbb{B}(\bar{x}, \varepsilon)$:

$$\begin{array}{rcl} \pi_1(\bar{x}, \bar{y}) & \geq & \pi_1(x, \bar{y}) \\ \sum_{i \in I_{\bar{x}|\bar{y}}} (d(p_i, \bar{y}) - d(p_i, \bar{x}))q_i & \geq & \sum_{i \in I_{x|\bar{y}}} (d(p_i, \bar{y}) - d(p_i, x))q_i \\ \sum_{i \in I_{x|\bar{y}}} (d(p_i, \bar{y}) - d(p_i, \bar{x}))q_i & \geq & \sum_{i \in I_{x|\bar{y}}} (d(p_i, \bar{y}) - d(p_i, x))q_i \\ \sum_{i \in I_{x|\bar{y}}} d(p_i, x)q_i & \geq & \sum_{i \in I_{x|\bar{y}}} d(p_i, \bar{x})q_i \end{array}$$

Consequently, $\bar{x}$ is a local minimum of the convex positive function $\Phi(z) = \sum_{i \in I_{x|\bar{y}}} d(p_i, z)q_i$, $z \in \mathbb{R}^2$, in $\mathbb{B}(\bar{x}, \varepsilon)$. Then $\bar{x}$ is a global minimizer of $\Phi(z)$ and thus a Weber point of $I_{x|\bar{y}}$. $\square$

From Proposition 8 the following corollary can be obtained:

Corollary 9 *If $(\bar{x}, \bar{y})$ is a local equilibrium for P, then $(\bar{x}, \bar{y}) \in W \times W$. Moreover, $\bar{x}$ is a Weber point of $I_{\bar{x}|\bar{y}}$ and $\bar{y}$ is a Weber point of $I_{\bar{y}|\bar{x}}$.*

The above corollary reduces the search for local equilibrium pairs, and therefore the search for global equilibrium pairs, to Weber pair of points. Notice that a Weber pair is determined by a partition of the plane induced by a straight line. If the points in a candidate subset J are not collinear, then there is only one Weber point corresponding to J, but there may exist a segment of Weber points corresponding to J if the points in J are collinear (see [24]). Hence an infinite number of Nash equilibria may exist as shown in an example of subsection 5.1.

It is easy to see that the inverse of Corollary 9 is not true in general. In Figure 1, a partition induced by the dashed line is given. The pair of Weber points (w_1, w_2) corresponding to the partition, is not a local equilibrium for P. Indeed, the perpendicular bisector of the segment $[w_1, w_2]$ gives a different partition. On the contrary, if the partition induced by a pair of Weber points fits the original partition, then the Weber pair might be a local equilibrium pair.

The following proposition gives a complete characterization of local equilibria for P.

Proposition 10 *The pair $(\bar{x}, \bar{y})$ is a local equilibrium for P if and only if:*

(a) $\bar{x}$ is a Weber point of $I_{x|\bar{y}}$ for all x such that $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$, and

(b) $\bar{y}$ is a Weber point of $I_{y|\bar{x}}$ for all y such that $I_{\bar{y}|\bar{x}} \subseteq I_{y|\bar{x}} \subseteq I_{\bar{y}|\bar{x}} \cup I_{\bar{x}\bar{y}}$.

Proof. If $(\bar{x}, \bar{y})$ is a local equilibrium for P, then conditions (a) and (b) hold by Proposition 8. Conversely, suppose that the pair $(\bar{x}, \bar{y})$ satisfies conditions (a) and (b). By Lemma 6, there exists

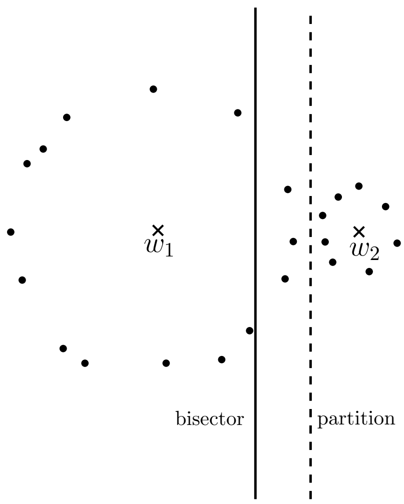

Figure 1: Two Weber points w_1 and w_2 which are not a local equilibrium pair.

$\varepsilon_1 > 0$ such that $I_{\bar{x}|\bar{y}} \subseteq I_{x|\bar{y}} \subseteq I_{\bar{x}|\bar{y}} \cup I_{\bar{x}\bar{y}}$ for all $x \in \mathbb{B}(\bar{x}, \varepsilon_1)$. Then:

$$\begin{aligned}\pi_1(x,\bar{y}) &= \sum_{i \in I_{x|\bar{y}}} (d(p_i,\bar{y}) - d(p_i,x))q_i \\ &\leq \sum_{i \in I_{x|\bar{y}}} (d(p_i,\bar{y}) - d(p_i,\bar{x}))q_i \\ &= \sum_{i \in I_{\bar{x}|\bar{y}}} (d(p_i,\bar{y}) - d(p_i,\bar{x}))q_i \\ &= \pi_1(\bar{x},\bar{y})\end{aligned}$$

In a similar way, we can prove that there exists $\varepsilon_2 > 0$ such that $\pi_2(\bar{x}, y) \leq \pi_2(\bar{x}, \bar{y})$ for all $y \in \mathbb{B}(\bar{y}, \varepsilon_2)$. By taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ it follows that $(\bar{x}, \bar{y})$ is a local equilibrium pair. □

Figure 2 illustrates Proposition 10. The demand points are represented as solid dots. The demand point $\bar{x}$ (resp. $\bar{y}$) satisfies condition (a) (resp. (b)) of Proposition 10, and the pair $(\bar{x}, \bar{y})$ is a local equilibrium. Note that if we remove $\bar{y}$, then $(\bar{x}, \bar{y})$ is not a local equilibrium because $\bar{y}$ is not a Weber point of $I_{\bar{y}|\bar{x}}$.

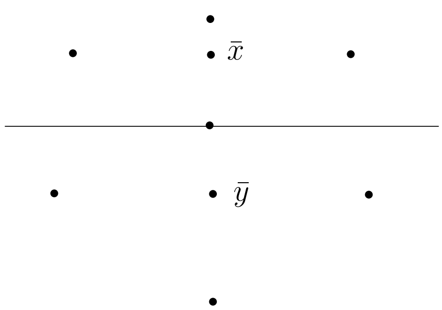

Figure 2: The pair $(\bar{x}, \bar{y})$ is a local equilibrium and satisfies the two conditions in Proposition 10.

The following result provides a procedure for finding the Nash equilibria of a demand point set P.

Proposition 11 *The pair* $(\bar{x}, \bar{y})$ *is a global equilibrium for* P *if and only if* $\pi_1(\bar{x}, \bar{y}) \geq \pi_1(w, \bar{y})$ *and* $\pi_2(\bar{x}, \bar{y}) \geq \pi_2(\bar{x}, w)$ *for all* w *in* W*.*

Proof. The necessary condition is straightforward. To see the sufficient condition we prove only that $\pi_1(\bar{x}, \bar{y}) \geq \pi_1(x, \bar{y})$ for all x in $\mathbb{R}^2$. The proof for π_2 is analogous.

Let x be a point in $\mathbb{R}^2$. Notice that, based on Proposition 1, the function $\pi_1(x, \bar{y})$ is continuous in $\mathbb{R}^2$ and takes value zero when $\|x\|$ is big enough. Then a maximum value of the function $\pi_1(x, \bar{y})$ is reached. Moreover, from Proposition 8, all local maxima of $\pi_1(x, \bar{y})$ are Weber points. Hence, the point $\bar{x}$ is a Weber point satisfying:

$$\pi_1(\bar{x}, \bar{y}) = \max\{\pi_1(w, \bar{y}) : w \in W\} \geq \pi_1(x, \bar{y}), \quad \forall x \in \mathbb{R}^2$$

and the result follows. □

5 The algorithm

The properties given in the above section suggest a procedure in order to find all pure strategy Nash equilibria in the location game. The idea of our procedure, stated in Algorithm 1, is the following. In a first step, we find all local equilibrium pairs corresponding to partitions of P induced by a straight line. After that, by using Proposition 11, we test if each local equilibrium pair found is or not global.

By considering Algorithm 1, we arrive to the following result:

Proposition 12 *An approximation to all the global equilibrium pairs for a demand set* P *with* n *points in the plane can be found in* $O(n^5)$ *time.*

Proof. The correctness of Algorithm 1 is due to Propositions 10 and 11. The complexity is the following.

Finding all partitions of P induced by a straight line can be done in $O(n^3)$ time by using the dual arrangement of P. Refer to [11].

Computing the set of Weber points of a point set (lines 4 and 5) can be done by using Weiszfeld's algorithm [29, 30]. In order to analyze the complexity we assume that Weiszfeld's algorithm makes a sufficient large but constant number of iterations. Thus its time complexity is linear. Moreover, we can also use the $O(n)$-time randomized (or $O(n \log n)$-time deterministic) ε-approximation algorithm of [6] to compute the Weber point of a point set.

Discarding local equilibrium pairs in line 8 can be done in linear time if w_2 is a point. In case w_2 is segment, we partition w_2 into $O(n)$ subsegments so that in each subsegment s the partition of P induced by the bisector of $[w_1, x]$ does not change for all $x \in s$. We select the subsegment s of w_2 such that the pair (w_1, x) satisfies Proposition 10 for all $x \in s$, and set w_2 equal to s. This step can be done in $O(n \log n)$ time by considering the order of the subsegments of w_2. We proceed analogously in order to discard local equilibrium pairs in line 11.

A subset of P with exactly k points, whose set of Weber points is a segment and belongs to W, is a k-set [7] of P consisting of collinear points. Notice that every collinear k-set of P, $k \geq 3$, induce a 2-set of P which is also collinear. Then, the number of collinear k-sets, $k \geq 2$, is at most twice the number of 2-sets, and thus $O(n)$ [7]. Denote by n_p (resp. n_s) the number of pairs in A such that both elements are points (resp. one element is a segment). We have that n_p is $O(n^2)$ and n_s

Algorithm 1 Compute all global equilibrium pairs
Input: A demand point set P.
Output: All global equilibrium pairs of P.

$A \leftarrow \emptyset$
$W \leftarrow \emptyset$
$\{$Find the set A of all local equilibrium pairs:$\}$
for all partitions $\{J_1, J_2\}$ of P induced by a straight line **do**
    $w_1 \leftarrow$ the set of Weber points of J_1
    $w_2 \leftarrow$ the set of Weber points of J_2
    $W \leftarrow W \cup \{w_1, w_2\}$
    if w_1 is a point **then**
        $w_2 \leftarrow \{w \in w_2 : (w_1, w)$ is a local equilibrium pair$\}$
        $A \leftarrow A \cup \{(w_1, w_2)\}$
    else if w_2 is a point **then**
        $w_1 \leftarrow \{w \in w_1 : (w, w_2)$ is a local equilibrium pair$\}$
        $A \leftarrow A \cup \{(w_1, w_2)\}$
    end if
end for
$\{$Discard all local equilibrium pairs which are not global:$\}$
for all pairs (w_1, w_2) in A **do**
    for all sets u of Weber points in W **do**
        if w_1 is a point **then**
            Remove from w_2 the set $\{w \in w_2 : \exists u' \in u$ such that $\pi_1(w_1, w) < \pi_1(u', w)\}$
            Remove from w_2 the set $\{w \in w_2 : \exists u' \in u$ such that $\pi_2(w_1, w) < \pi_2(w_1, u')\}$
        else
            Remove from w_1 the set $\{w \in w_1 : \exists u' \in u$ such that $\pi_1(w, w_2) < \pi_1(u', w_2)\}$
            Remove from w_1 the set $\{w \in w_1 : \exists u' \in u$ such that $\pi_2(w, w_2) < \pi_2(w, u')\}$
        end if
        if $w_1 = \emptyset$ **or** $w_2 = \emptyset$ **then**
            $A \leftarrow A \setminus \{(w_1, w_2)\}$
        end if
    end for
end for
return A

is $O(n)$. Thus, the set W is obtained in $O(n^3)$ time and the set A in $n_p \cdot O(n) + n_s \cdot O(n \log n) = O(n^2) \cdot O(n) + O(n) \cdot O(n \log n) = O(n^3)$ time.

In line 18 local equilibrium pairs belonging to $w_1 \times w_2$ are discarded, where w_1 is a point. This can be done by considering four cases:

Case 1: Both w_2 and u are points. In linear time we compute and compare the profits $\pi_1(w_1, w_2)$ and $\pi_1(u, w_2)$.

Case 2: w_2 is a point and u is a segment. First, we find in $O(n \log n)$ time a point $u' \in u$ such that P_u is equal to $I_{u'|w_2}$, where P_u is the subset of P for which u is the set of Weber points. After that, we compute and compare the profits $\pi_1(w_1, w_2)$ and $\pi_1(u', w_2)$.

Case 3: w_2 is a segment and u is a point. First we find in $O(n \log n)$ time the maximal-length subsegment w_2' of w_2 such that P_{w_2} is equal to $I_{u|w}$ for all $w \in w_2'$. After that, we discard the points $w \in w_2'$ such that $\pi_1(w_1, w) < \pi_1(u, w)$ by comparing the convex functions $\pi_1(w_1, x)$ and $\pi_1(u, x)$, $x \in w_2'$.

Case 4: Both w_2 and u are segments. We find in $O(n \log n)$ time the maximal-length subsegment w_2' of w_2 such that for all $w \in w_2'$ there exists a point $u' \in u$ satisfying $P_u = I_{u'|w}$. After that, we pick any element $u' \in u$ and proceed similarly as was done in *Case 3*.

A similar processing divided into four cases can be done for lines 19, 21, and 22. In the overall algorithm, we spend $n_p \cdot n_p \cdot O(n) = O(n^5)$ time in Case 1, $n_p \cdot n_s \cdot O(n \log n) = O(n^4 \log n)$ time in Case 2, $n_s \cdot n_p \cdot O(n \log n) = O(n^4 \log n)$ time in Case 3, and $n_s \cdot n_s \cdot O(n \log n) = O(n^3 \log n)$ time in Case 4. Thus, Algorithm 1 runs in $O(n^5)$ time. □

Remarks:

(1) In Algorithm 1 we do not consider the case in which both w_1 and w_2 (computed in lines 4 and 5) are segments. Notice that in this situation the demand points are partitioned by a straight line into two sets of collinear points, and it almost never occurs in practice. If this case happens, it can be treated as follows: First, we discard from w_2 the points w_2' such that there exists a set of Weber points $u \in W \setminus \{w_1, w_2\}$ containing a point u' that satisfies $P_u = I_{u'|w_2'}$ and $\pi_1(w_1, w_2') < \pi_1(u', w_2')$. After that, we discard from w_1 the points w_1' such that there exists a set of Weber points $u \in W \setminus \{w_1, w_2\}$ containing a point u' that satisfies $P_u = I_{u'|w_1'}$ and $\pi_2(w_1', w_2) < \pi_2(w_1', u')$. Notice that w_1 (resp. w_2) is now equal to the union of at most two maximal-length segments. Finally, we discard from $w_1 \times w_2$ the elements that do not satisfy Proposition 10, and the remaining pair of points are thus global equilibria.

(2) Notice that the time complexity of Algorithm 1 strongly depends on the number k of elements of the set A. We considered in the time complexity analysis that k is $O(n^2)$, but maybe it is too much. In some real situations demand points are well distributed and we can observe that any line that partitions the demand points into two subsets, one of which containing many less elements than the other, gives with high probability pairs of Weber points that do not correspond to local equilibrium pairs.

(3) If we are interested in finding just one global equilibrium pair, Proposition 4 can be used to find it more efficiently. In fact, the local equilibrium pair of A with minimum social cost is a global equilibrium pair and, as a consequence, a global equilibrium pair for n demand points in the plane can be found in $O(n^3)$ time.

(4) In [5] it is shown that in general there is no exact algorithm involving only arithmetic operations and kth roots for computing the Weber points of a point set. Therefore only numerical

or symbolic approximations are possible under this model of computation. In this sense Algorithm 1 is an approximation algorithm, unless a different model of computation is used.

5.1 Examples

In the following, we show some examples which illustrate some results related to Nash equilibria of the proposed game.

Example 1. The inverse of Proposition 4 is not true in general, as we mentioned in Section 3. Let us give a counterexample. Consider the demand set

$$P = \{p_1 = (2.3038, 2.5668), p_2 = (2.2868, 1.3156), p_3 = (2.2534, 0.0346), p_4 = (2.0611, 1.4695),$$
$$p_5 = (4.5080, 2.6378), p_6 = (0.0279, 2.7786), p_7 = (1.4870, 3.2199), p_8 = (0.2458, 2.9565),$$
$$p_9 = (3.4659, 1.2871), p_{10} = (3.2505, 2.1728), p_{11} = (4.9149, 2.5595), p_{12} = (2.7634, 0.6786),$$
$$p_{13} = (2.0004, 3.1668), p_{14} = (0.9939, 1.9922), p_{15} = (3.1260, 2.2113)\}.$$

The demand at each one of these points (filled circles in Figure 3) is considered equal to one.

The global equilibria (stars in Figure 3) have been obtained by applying Algorithm 1 . Table 1 shows the local equilibria together with their social cost, arranged in increasing order. We denote by w_x (resp. w_y) the abscissa (resp. ordinate) of w, and by (w, w') a local equilibrium pair.

The global equilibrium pairs illustrated in Figure 3 are: (w_1, w'_1), (w_2, w'_2) and (w_3, w'_3). Note that, there are infinite global equilibria that correspond to all pairs (w_3, w) such that w is any point in the segment joining p_5 and p_{11}, in Figure 3. We also observe in Table 1 that only the first pair of points w_1=(0.8512, 2.8212), w'_1=(3.0357, 1.7817) is a global equilibrium that minimizes the social cost function.

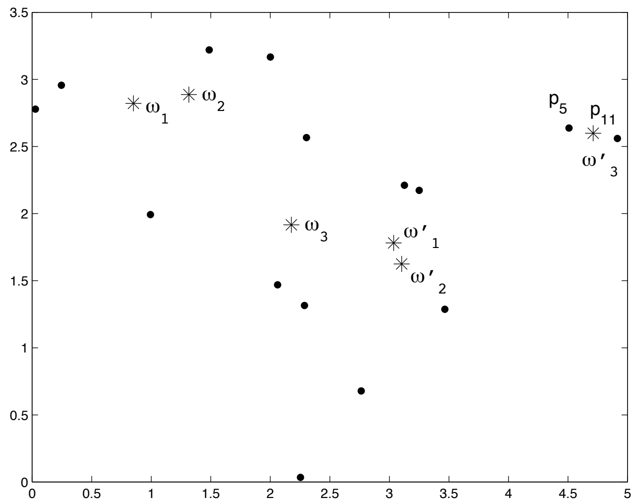

Figure 3: Example 1.

Example 2. It is possible the existence of a unique local equilibrium which is the global equilibrium.

	Local equilibria				
Global Equilibria	w_x	w_y	w'_x	w'_y	Social Cost
(w_1, w'_1)	0.8512	2.8212	3.0357	1.7817	15.5433
(w_2, w'_2)	1.3162	2.8870	3.1027	1.6252	15.6237
	0.3162	2.8658	2.9978	1.9259	15.7090
	3.2466	2.1792	1.8695	1.5623	17.2673
(w_3, w'_3)	2.1771	1.9168	4.7115	2.5987	17.3665
	1.9010	2.0390	4.3455	2.4272	17.5712
	2.3038	2.5668	2.5082	1.0225	17.7081
	2.3247	2.6008	2.2868	1.3156	17.8206

Table 1: The global and local equilibria for P in Example 1.

For instance, consider the following 20 demand points set:

$$P = \{p_1 = (0.9501, 0.7621), p_2 = (0.2311, 0.4565), p_3 = (0.6068, 0.0185), p_4 = (0.4860, 0.8214), p_5 = (0.8913, 0.4447), p_6 = (3.8132, 3.2722), p_7 = (3.0099, 3.1988), p_8 = (3.1389, 3.0153), p_9 = (3.2028, 3.7468), p_{10} = (3.1987, 3.4451), p_{11} = (3.6038, 3.9318), p_{12} = (4.3028, -1.1002), p_{13} = (4.5417, -1.1784), p_{14} = (4.1509, -1.3551), p_{15} = (4.6979, -1.1820), p_{16} = (4.3784, -1.3398), p_{17} = (4.8600, -1.6580), p_{18} = (4.8537, -1.7103), p_{19} = (4.5936, -1.6588), p_{20} = (4.4966, -1.4659)\}.$$

The set of points P is represented in Figure 4 and the demand at each point is considered equal to one. We have three clusters, distributed around the points $(0.5, 0.5)$, $(3.5, 3.5)$ and $(4.5, -1.5)$. We found 190 Weber pairs but only one local equilibrium (represented with triangles in Figure 4).

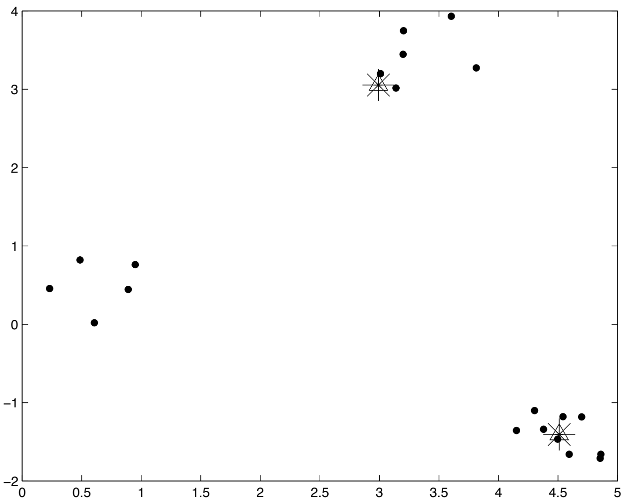

Figure 4: Example 2.

Table 2 shows the local equilibrium together with its social cost.

Example 3. Consider a new unweighted point set by removing the last five points from the demand set in Example 2. In this case, we found three local equilibria and only one of these is a global equilibrium.

Table 3 shows the results and Figure 5 illustrates the location of local (triangles) and global (stars) equilibrium pairs.

	Local equilibria				
Global Equilibria	w_x	w_y	w'_x	w'_y	Social Cost
(w_1, w'_1)	2.9920	3.0532	4.5097	-1.4058	23.4304

Table 2: There only exists one local equilibrium in Example 2.

	Local equilibria				
Global Equilibria	w_x	w_y	w'_x	w'_y	Social Cost
(w_1, w'_1)	3.1988	3.4451	1.0454	0.3492	19.9090
	2.9920	3.0532	4.5416	-1.1784	21.6474
	3.0227	3.1600	4.3142	-1.1241	21.7083

Table 3: The local equilibria in Example 3.

6 Conclusions

In this paper we have presented a competition location model that can be viewed as a geometric location game in the two-dimensional Euclidean space. The goal is to find the equilibrium positions for two firms that select their locations and then set delivered prices in order to maximize their profits. In the existing literature, Nash equilibria to this type of game are found by social cost minimization. We have presented characterizations of both a local and a global Nash equilibrium and proved that there may exist Nash equilibria which are not social cost minimizers. Based on such characterizations, we have proposed an algorithm to find all possible Nash equilibria. An advantage of the approach is that the method can be easily adapted to a more general model by using a measuring distance different to the Euclidean one. Note that in this case, the only difference is that the bisector of two points is not a straight line and the computational model has to be augmented. Similarly, the results presented in this setting can be extended to the case in which the marginal delivered cost is given by the production cost (independent of distance) plus the delivered cost (proportional to distance). As an open problem, an improvement of the time complexity for computing the Nash equilibrium pairs is of interest.

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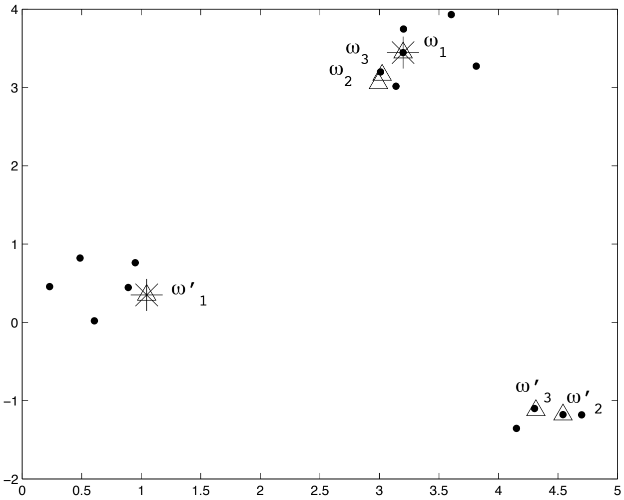

Figure 5: Example 3.

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